A METHOD FOR CONSTRUCTING ε-VALUE FUNCTIONS FOR THE BOLZA PROBLEM OF OPTIMAL CONTROL

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The problem considered is that of approximate minimisation of the Bolza problem of optimal control. Starting from Bellman's method of dynamic programming, we define the ε-value function to be an approximation to the value function being a solution to the Hamilton-Jacobi equation. The paper shows an approach that can be used to construct an algorithm for calculating the values of an ε-value function at given points, thus approximating the respective values of the value function.

Keywords: non-linear optimisation, Bolza problem, optimal control, Hamilton-Jacobi equation, dynamic programming, value function, approximate minimum

1. Introduction

The aim of this paper is to provide an effective numerical algorithm for finding an ε-value function for the Bolza optimal control problem. The ε-value function is a step towards a numerical algorithm for finding optimal control for the Bolza problem in quite a general setting. Our approach is similar to the one presented in (Jacewicz, 2001).

We consider the problem of finding optimal control for the following problem, known as the Bolza problem:

$$\text{minimize } J(x,u) = \int_a^b L\big(t, x(t), u(t)\big)\,\mathrm{d}t + l\big(x(b)\big), \quad (1)$$

where $x : [a,b] \to \mathbb{R}^n$ is an absolutely continuous function and $u : [a,b] \to \mathbb{R}^m$ is a Lebesgue measurable function. The functional $J(x,u)$ is called the cost, the function $x(\cdot)$ is called the trajectory, and the function $u(\cdot)$ is called the control. Both functions are subject to the following constraints:

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) \text{ a.e. in } [a,b], \quad (2)$$

$$u(t) \in U, \quad t \in [a,b], \quad (3)$$

$$x(a) = c, \quad (4)$$

where the functions $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $l : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are given in the problem, $U \subset \mathbb{R}^m$ is an open set, and c is a point in $\mathbb{R}^n$.

We additionally assume that the functions L, f and l satisfy

(Z)
- (a) $(t,x,u) \to L(t,x,u)$ and $(t,x,u) \to f(t,x,u)$ satisfy on $[a,b] \times \mathbb{R}^n \times U$ the Lipschitz condition with respect to the compound variable (t,x,u).
- (b) $x \to l(x)$ satisfies locally the Lipschitz condition in $\mathbb{R}^n$

Definition 1. A *pair* of functions $(x(\cdot), u(\cdot))$ is called *admissible* when it satisfies (2), (3) and the function $t \to L(t, x(t), u(t))$ is integrable. The trajectory $t \to x(t)$ is then called an *admissible trajectory*, and the control $t \to u(t)$ is then called an *admissible control*.

Now we can restate the Bolza problem: Find $\inf J(x,u)$, where the infimum is taken over all admissible pairs $x(\cdot)$, $u(\cdot)$ satisfying (4).

Definition 2. Any admissible control, for which the minimum of the functional $J(\cdot,\cdot)$ is reached, is called *optimal control* and denoted by $u_{\min}(\cdot)$.

An exact solution to the Bolza problem is rather hard to find. However, we can consider the problem of finding an approximate solution, i.e., such admissible pairs $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$, defined on $[a,b]$, for which $x_\varepsilon(a) = c$, and the following condition holds:

$$J(x_\varepsilon, u_\varepsilon) \le \inf J(x,u) + \varepsilon(b-a), \quad (5)$$

where $\varepsilon > 0$ is some real number. Every control $u_\varepsilon(\cdot)$ obtained in this way is called an ε-*optimal control*.

In this paper we are going to approximate the Bolza problem with methods of dynamic programming. Such an approach is extensively described in the existing literature (Cesari, 1983; Fleming and Rishel, 1975). However, unlike most currently developed methods for solving Hamilton-Jacobi equations (and thus the Bolza problem), which combine the direct use of the classical approaches with clever numerical approximations, we use a distinct method developed in (Jacewicz, 2001). Recent publications in the field, which rely on the classical approach with innovative numerical solutions, include (Karlsen and Risebro, 2002), where the authors use a front tracking method developed for dealing with hyperbolic conservation laws to tackle Hamilton-Jacobi equations; (Kurganov and Tadmor, 2000), where another method originally developed for hyperbolic conservation laws is used, namely, the semi-discrete central schemes; (Bryson and Levy, 2001), where also central schemes are used; (Tang *et al.*, 2003), where an adaptive mesh instead of a rectangular one is used for solving the H-J equation numerically. All those papers are focused on obtaining practical numerical solutions to the H-J equation, while other considerations, such as convergence or stability, are secondary. Among the few papers which approach the subject in a different manner there is the article (Szpiro and Dupuis, 2002), where the main emphasis is on the proof of the convergence of the proposed method. Szpiro and Dupuis develop a novel, probabilistic approach to H-J equations. However, the method proposed there is rather complicated. Our method is tailored to the Bolza problem and, in our belief, it is much simpler. It also has an interesting property—the result is being calculated with precision given *a priori*. The approximation is stable and convergent.

Two most important contributions of this paper are the elimination of some artificial conditions that impose severe limitations on the acceptable class of functions $L(\cdot,\cdot,\cdot)$, $f(\cdot,\cdot,\cdot)$, which are stated in (Jacewicz, 2001) (Lemma 5) at the stage of constructing the approximate value function and an effective, easily adaptable to machine implementation, algorithm for constructing such approximated value functions. These limitations, expressed in the assumptions (L1)–(L3) of Lemma 5 (p. 415), have the following implications:

- (L1) requires that the first derivative of the value function with respect to x be non-zero inside any of the sets into which the domain is partitioned;
- (L2) requires that the graph of the function $f(\cdot,\cdot,\cdot)$ must lie between the graphs of two functions linear with respect to u;
- (L3) has some consequences limiting the allowed class of functions $L(\cdot,\cdot,\cdot)$.

Because of those limitations, Jacewicz's method can be considered only theoretical. It is very hard to find a real-world problem that would fit into such limitations. Additionaly, the proofs of theorems and numerical calculations are also complicated, the latter because the choice of starting functions is limited by the assumption (L1). The current paper does not necessitate all those assumptions. It also simplifies the notation which makes the numerical algorithm shorter and easier to apply. However, in cases where Jacewicz's method is applicable, the method presented here gives substantially the same results. Therefore, we did not find it necessary to present a comparison of those two methods on the same example.

The paper is further structured as follows: Section 2 ends with the fundamental theorem called the verification theorem. It specifies conditions which have to be fullfilled by a function in order to be an ε-value function. In Section 3 we specify a method for constructing a function that is a suitable candidate to be an ε-value function. Section 4 shows how to calculate the values of an ε-value function at specified points. It also includes an example.

2. Definition and Properties of a Value Function and an Approximate Value Function

2.1. Dynamic Programming—the Value Function

Let $T \subset [a,b] \times \mathbb{R}^n$ be a set with non-empty interior, covered by graphs of admissible trajectories, i.e., for every $(t_0, x_0) \in T$ there exists an admissible pair $x(\cdot)$, $u(\cdot)$, defined on $[t_0, b]$, such that $x(t_0) = x_0$ and $(s, x(s)) \in T$ for $s \in [t_0, b]$. The assumption that the interior of T is non-empty is essential for further deliberations and constitutes some limitation of this method.

Definition 3. A function $(t,x) \to S(t,x)$ defined in T is called a *value function* when

$$S(t,x) = \inf \left\{ \int_t^b L\big(s, x(s), u(s)\big)\,\mathrm{d}s + l\big(x(b)\big) \right\}, \tag{6}$$

where the infimum is taken over all admissible trajectories $s \to x(s)$, $s \in [t,b]$, which start from $(t,x) \in T$, $x(t) = x$, and their graphs are contained in T.

2.2. Dynamic Programming—the Approximate Value Function

We will now discuss the approximation of the value function by an ε-value function. However, because we modify the definition of the ε-value function (compared with (Jacewicz, 2001)) and of the ε-optimal trajectory, the proof of the verification theorem (Thm. 1) will also be presented.

Definition 4. A function $(t,x) \to S_\varepsilon(t,x)$ defined on the set T is called an ε-*value function* iff

$$S(t,x) \le S_\varepsilon(t,x) \le S(t,x)+\varepsilon(b-a),\ (t,x) \in T, \quad (7)$$

$$l(x) \le S_\varepsilon(b,x) \le l(x)+\varepsilon(b-a),\ \ (b,x) \in T,$$

where $(t,x) \to S(t,x)$ is a value function, $x \to l(x)$ is a function described in the formulation (1)–(4) of the Bolza problem that satisfies (Z), and $\varepsilon > 0$ is a fixed number, which will be assumed to be constant in all further deliberations.

Obviously, for a given $\varepsilon > 0$, the above ε-value function is not uniquely defined, and therefore we speak of many ε-value functions.

One should notice that $(t,x) \to S_\varepsilon(t,x)$ is finite in T.

Definition 5. An admissible trajectory $s \to x_\varepsilon(s)$, $s \in [t,b]$, $x_\varepsilon(t) = x$ is called ε-*optimal* if for all admissible trajectories $s \to x(s)$, $s \in [t,b]$, $x(t) = x$ we have

$$\begin{aligned}\int_t^b L\big(s,x(s),u(s)\big)\mathrm{d}s + l\big(x(b)\big) + \varepsilon(b-t)&\\ \ge \int_t^b L\big(s,x_\varepsilon(s),u_\varepsilon(s)\big)\mathrm{d}s + l\big(x_\varepsilon(b)\big).&\quad (8)\end{aligned}$$

An ε-optimal trajectory always exists because every set of real numbers has an infimum.

We are now ready to formulate the fundamental theorem of the introductory part:

Theorem 1. *Let $T \subset [a,b]\times\mathbb{R}^n$ be an open set, $S(t,x)$ be a value function in T, and $(t,x) \to G(t,x)$ be a function defined on T, which is almost everywhere on T a $C^1(T)$ solution of the following inequality:*

$$-\frac{\varepsilon}{2} \le G_t(t,x)+\min_{u\in U}\big\{G_x(t,x)f(t,x,u)+L(t,x,u)\big\} \le 0 \quad (9)$$

satisfying the boundary condition $l(x) \le G(b,x) \le l(x)+\frac{\varepsilon}{2}(b-a), (b,x) \in T$.

Moreover, if $x_\varepsilon(\cdot)$, $u_\varepsilon(\cdot)$ is an admissible pair defined on $[t_1,b]$, $a \le t_1 \le b$, $(t_1, x_\varepsilon(t_1)) \in T$, such that for almost every $\bar t \in [t_1,b]$ we have

$$\begin{aligned}-\frac{\varepsilon}{2} \le G_t\big(\bar t,x_\varepsilon(\bar t)\big) + G_x\big(\bar t,x_\varepsilon(\bar t)\big)f\big(\bar t,x_\varepsilon(\bar t),u_\varepsilon(\bar t)\big)&\\ + L\big(\bar t,x_\varepsilon(\bar t),u_\varepsilon(\bar t)\big) \le 0,&\quad (10)\end{aligned}$$

then $x_\varepsilon(\cdot)$ is an ε-optimal trajectory. Additionally, if for some point $(\tilde t,\tilde x) \in T$ there exists an admissible trajectory starting from $(\tilde t,\tilde x)$ and satysfying (10), then $G(\tilde t,\tilde x) = S_\varepsilon(\tilde t,\tilde x)$ where $S_\varepsilon(\cdot,\cdot)$ is some ε-value function.

Proof. Assume that $x(\cdot)$ is an admissible trajectory. Obviously, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}G\big(t,x(t)\big) + L\big(t,x(t),u(t)\big)\\ &\quad= G_t\big(t,x(t)\big) + G_x\big(t,x(t)\big)f\big(t,x(t),u(t)\big)\\ &\qquad+ L\big(t,x(t),u(t)\big) \ge -\frac{\varepsilon}{2}. \quad (11)\end{aligned}$$

Because (9) holds almost everywhere in T, for every admissible trajectory starting from (t,x) we obtain

$$G\big(t,x(t)\big) \le \int_t^b L\big(s,x(s),u(s)\big)\mathrm{d}s + G\big(b,x(b)\big) + \frac{\varepsilon}{2}(b-t). \quad (12)$$

Applying the boundary condition and the definition of the value function, we get

$$G(t,x) \le S(t,x) + \varepsilon(b-t) \quad (13)$$

for all $(t,x) \in T$. From (10) we have that along the trajectory $x_\varepsilon(\cdot)$, starting from $(\tilde t,\tilde x)$, the following holds:

$$\begin{aligned}&G\big(\tilde t,x_\varepsilon(\tilde t)\big) - \int_{\tilde t}^b L\big(s,x_\varepsilon(s),u_\varepsilon(s)\big)\mathrm{d}s - \frac{\varepsilon}{2}(b-\tilde t)\\ &\quad\le G\big(b,x_\varepsilon(b)\big)\\ &\quad\le G\big(\tilde t,x_\varepsilon(\tilde t)\big) - \int_{\tilde t}^b L\big(s,x_\varepsilon(s),u_\varepsilon(s)\big)\mathrm{d}s. \quad (14)\end{aligned}$$

From (14) and the boundary condition we obtain

$$\begin{aligned}l\big(x_\varepsilon(b)\big) &\le G\big(b,x_\varepsilon(b)\big)\\ &\le G\big(\tilde t,x_\varepsilon(\tilde t)\big) - \int_{\tilde t}^b L\big(s,x_\varepsilon(s),u_\varepsilon(s)\big)\mathrm{d}s, \quad (15)\end{aligned}$$

which gives

$$\begin{aligned}S(\tilde t,\tilde x) &\le \int_{\tilde t}^b L\big(s,x_\varepsilon(s),u_\varepsilon(s)\big)\mathrm{d}s + l\big(x_\varepsilon(b)\big)\\ &\le G\big(\tilde t,x_\varepsilon(\tilde t)\big). \quad (16)\end{aligned}$$

This, combined with (13), proves that $x_\varepsilon(\cdot)$ is an ε-optimal trajectory and that $G(\tilde t,\tilde x) = S_\varepsilon(\tilde t,\tilde x)$. ■

3. Construction of ε-Value Functions and a Computer Algorithm

The method leading to the construction of ε-value functions was first described in (Jacewicz, 2001). However, our method does not need additional assumptions and is better suited to being used as a base for computer programs. In particulars, we do not assume anything about the functions $(t,x,u) \to f(t,x,u)$ and $(t,x,u) \to L(t,x,u)$ that goes beyond assumptions listed in (Z) in the formulation of the Bolza problem and those additional assumptions that are the consequence of the verification theorem.

3.1. Construction of ε-Value Functions

Let $T \subset [a,b] \times \mathbb{R}^n$ be a compact set with a non-empty interior covered with graphs of admissible trajectories and let $U \subset \mathbb{R}^m$ be a compact set.

We will begin the construction of the ε-value function by choosing some arbitrary $C^2(T)$ function $(t,x) \to w(t,x)$ that satisfies the boundary condition $w(b,x) = l(x) + \varepsilon/2,\ (b,x) \in T$.

We will define on T a function $(t,x) \to F(t,x)$ that will correspond to the right-hand side of the Hamilton-Jacobi equation:

$$F(t,x) := \frac{\partial}{\partial t} w(t,x) + \min_{u \in U} \left\{ \frac{\partial w}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\}, \tag{17}$$

where the functions $(t,x,u) \to f(t,x,u)$ and $(t,x,u) \to L(t,x,u)$ satisfy the conditions (Z), the function $w(\cdot,\cdot)$ is defined as above and the infimum is replaced with the minimum due to the compactness of U.

The function $(t,x) \to F(t,x)$ is continuous on T. Moreover, it satisfies the Lipschitz condition on T.

Since T is a compact set, the function $F(\cdot,\cdot)$ reaches its bounds on T, which we denote by κ_d and κ_g, respectively,

$$\kappa_d \le F(t,x) \le \kappa_g \qquad \text{for all } (t,x) \in T. \tag{18}$$

The function $F(\cdot,\cdot)$ defined above has in T values of different signs and therefore it cannot satisfy (9). In order to find a function that satisfies the assumptions of the verification theorem, we will now define a family of functions $(t,x) \to F_1^k(t,x),\ k \in \mathbb{N}$. These functions will satisfy for all $k > k_{\frac{\varepsilon}{2}}$ the inequality (9), where $k_\varepsilon \in \mathbb{N}$ are numbers that depend on the chosen ε, such that $k_\varepsilon \to \infty$ for $\varepsilon \to 0$. The function $F_1^k(\cdot,\cdot)$ for every k is described by the following formula and the construction of $(t,x) \to w_1^k(t,x),\ k \in \mathbb{N}$, is described below:

$$F_1^k(t,x) := \frac{\partial}{\partial t} w_1^k(t,x) + \min_{u \in U} \left\{ \frac{\partial w_1^k}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\} \tag{19}$$

for every $(t,x) \in T$. We will begin the construction of $w_1^k(\cdot,\cdot)$ by defining its domain. Let us divide the interval $[\kappa_d, \kappa_g] \subset \mathbb{R}$, being the image of the set T in the mapping $(t,x) \to F(t,x)$, creating a k subinterval $[y_i, y_{i+1}],\ i \in \{1,\dots,k\}$, such that $\kappa_d = y_1 < y_2 < \dots < y_{k+1} = \kappa_g$, and that for all $i \in \{1,\dots,k\}$ we have $|y_{i+1} - y_i| = \frac{1}{k}|\kappa_g - \kappa_d|$. Write $\eta_k := \frac{1}{k}|\kappa_g - \kappa_d|$.

Now we divide the set T into the following subsets $P_j^k,\ j \in \{1,\dots,k\}$:

$$P_1^k := \{(t,x) \in T : y_1 \le F(t,x) \le y_2\}, \tag{20}$$

$$P_j^k := \{(t,x) \in T : y_j < F(t,x) \le y_{j+1}\}, \quad j \in \{2,\dots,k\}. \tag{21}$$

The sets $P_j^k,\ j \in \{1,\dots,k\}$ constitute a covering of the set T, i.e., for every $i,j \in \{1,\dots,k\}, i \ne j, P_i^k \cap P_j^k = \emptyset$, and $\bigcup_{j=1}^k P_j^k = T$.

We will now define the auxiliary functions $(t,x) \to w_{1,j}^k(t,x)$ and $(t,x) \to F_{1,j}(t,x)$ on the sets $P_j^k,\ j \in \{1,\dots,k\}$, as follows:

$$w_{1,j}^k(t,x) := w(t,x) + y_{j+1}(b-t), (t,x) \in P_j^k, \tag{22}$$

$$F_{1,j}^k(t,x) := \frac{\partial}{\partial t} w_{1,j}^k(t,x) + \min_{u \in U} \left\{ \frac{\partial w_{1,j}^k}{\partial x}(t,x) f(t,x,u) + L(t,x,u) \right\}, \quad (t,x) \in P_j^k. \tag{23}$$

A simple calculation yields

$$F_{1,j}^k(t,x) = F(t,x) - y_{j+1}, \qquad (t,x) \in P_j^k, \tag{24}$$

which means

$$-\eta_k \le F_{1,j}^k(t,x) \le 0, \quad (t,x) \in P_j^k, \quad j \in \{1,\dots,k\}. \tag{25}$$

It is easy to notice that for some fixed $\varepsilon > 0$ we can always choose k_ε such that for every $m > k_\varepsilon$ we have $-\varepsilon \le F_{1,j}^m(t,x) \le 0$.

We define the function $w_1^k(\cdot,\cdot)$ (for fixed k) in $T = \bigcup_{j=1}^k P_j^k$ as follows:

$$w_1^k(t,x) := w_{1,j}^k(t,x) \text{ for } (t,x) \in P_j^k,\ j \in \{1,\dots,k\}. \tag{26}$$

Obviously, for every $k > k_{\frac{\varepsilon}{2}}$ the function $w_1^k(\cdot,\cdot)$ satisfies the inequality (9) of the verification theorem (Thm. 1) of the dynamic programming for some fixed $\varepsilon > 0$, and satisfies the boundary condition of this theorem (since its values for $t = b$ are equal to the corresponding values of $w(\cdot,\cdot)$), yet it is not a function of the class $C^2(T)$ (probably it is even a discontinuous function), and thus it does not fulfil the requirements. In order to satisfy the assumptions of the verification theorem, we have to smooth the function $w_1^k(\cdot,\cdot)$ by convoluting it with a function of the class $C^\infty(\mathbb{R}^{n+1})$ having compact support.

From now on we assume that k (the number of sets P_j^k) is a fixed natural number, $j \in \{1,\dots,k\}$, and $\beta > 0$ is some real number.

The function $\rho_\beta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of the class $C^\infty(\mathbb{R}^{n+1})$ having compact support, where $\beta \in \mathbb{R}_+$, is defined as follows:

Let $\rho_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^\infty(\mathbb{R}^{n+1})$ having a compact support, such that $\int_{\mathbb{R}^{n+1}} \rho_1(t,x)\mathrm{d}t\,\mathrm{d}x = 1$ and $\operatorname{supp}\rho_1 \subset B_1(\mathbb{R}^{n+1})$, where 'supp' denotes the support, and $B_\tau(\mathbb{R}^{n+1})$ for any $\tau \in \mathbb{R}$ is a ball in $\mathbb{R}^{n+1}$ with the center at 0 having the radius τ. Obviously, $\rho_\beta(t,x) := (1/\beta^{n+1})\rho_1(t/\beta, x/\beta)$. It is easy to see that such a function $\rho_\beta(\cdot,\cdot)$ is an infinitely smooth function having the compact support $\operatorname{supp}\rho_\beta \subset B_\beta(\mathbb{R}^{n+1})$ and $\int_{B_\beta(\mathbb{R}^{n+1})} \rho_\beta(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}^{n+1}} \rho_\beta(t,x)\,\mathrm{d}t\,\mathrm{d}x = 1$. An example of such a function will be given in the section devoted to the numerical algorithm.

Let us now define for each $\beta \in \mathbb{R}_+$ a new function $(t,x) \to w_2^{k,\beta}(t,x)$:

$$w_2^{k,\beta}(t,x) := (w_1^k * \rho_\beta)(t,x), \tag{27}$$

where the star denotes convolution.

From a theorem in (Adams, 1975) we have that for every k and β the function $w_2^{k,\beta}(\cdot,\cdot)$ is of the class $C^2(T)$, which means that the corresponding function $(t,x) \to F_2^{k,\beta}(t,x)$, defined by

$$\begin{aligned} F_2^{k,\beta}&(t,x) \\ &:= \frac{\partial}{\partial t} w_2^{k,\beta}(t,x) \\ &\quad + \min_{u\in U}\left\{\frac{\partial w_2^{k,\beta}}{\partial x}(t,x) f(t,x,u) + L(t,x,u)\right\}, \end{aligned} \tag{28}$$

is continuous in T.

We will now try to evaluate the function $F_2^{k,\beta}(\cdot,\cdot)$. Let Q be a set where $w_1^k(\cdot,\cdot)$ is discontinuous. The Lebesgue measure of Q in $\mathbb{R}^{n+1}$ is zero, which is a consequence of the definition of $w_1^k(\cdot,\cdot)$.

Lemma 1. *For every given, fixed k and for every $i \in \mathbb{N}$ there exists real $\hat\beta^{k,i} > 0$, such that for every $0 < \beta \le \hat\beta^{k,i}$ and for all $(t,x) \in T \backslash Q$ the following inequality is satisfied:*

$$\left|\frac{\partial}{\partial t} w_2^{k,\beta}(t,x) - \frac{\partial}{\partial t} w_1^k(t,x)\right| < \Big(1 + \frac{1}{i}\Big)\eta_k. \tag{29}$$

Proof. Take the arbitrary $(t,x) \in T\backslash Q$. Then for some $m \in \{1,\dots,k\}$ the point $(t,x) \in P_m^k$. Since $F(\cdot,\cdot)$ is uniformly continuous on T, we can always find $\hat\beta_1^k$ such that for all $(s,y) \in B_{\hat\beta_1^k}(\mathbb{R}^{n+1})$ we have the estimate $|F(t-s,x-y) - F(t,x)| < \frac{1}{2}\eta_k$. Therefore, we have one of the following cases, in accordance with the location of (t,x) in the set P_m^k:

Case 1: $\displaystyle\forall_{(s,y)\in B_{\hat\beta_1^k}(\mathbb{R}^{n+1})} \quad (t-s, x-y) \in P_m^k \cup P_{m-1}^k,$

Case 2: $\displaystyle\forall_{(s,y)\in B_{\hat\beta_1^k}(\mathbb{R}^{n+1})} \quad (t-s, x-y) \in P_m^k \cup P_{m+1}^k.$

We will give here the proof only for Case 1, since the proof for Case 2 is analogous.

Define

$D_\beta^m := \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s, x-y) \in P_m^k\right\}$,
$D_\beta^{m-1} := \left\{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s, x-y) \in P_{m-1}^k\right\}$.

Obviously, for every $0 < \beta < \hat\beta_1^k$ we have $B_\beta(\mathbb{R}^{n+1}) = D_\beta^m \cup D_\beta^{m-1}$, and $D_\beta^m \cap D_\beta^{m-1} = \emptyset$.

Thus

$$\begin{aligned} &\left|\frac{\partial}{\partial t} w_2^{k,\beta}(t,x) - \frac{\partial}{\partial t} w_1^k(t,x)\right| \\ &= \Big|\int_{B_\beta(\mathbb{R}^{n+1})} \frac{\partial}{\partial t} w_1^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad - \frac{\partial}{\partial t} w_1^k(t,x)\Big| \\ &= \Big|\int_{D_\beta^m} \frac{\partial}{\partial t} w_{1,m}^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + \int_{D_\beta^{m-1}} \frac{\partial}{\partial t} w_{1,m-1}^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad - \frac{\partial}{\partial t} w_1^k(t,x)\Big| \\ &= \Big|\int_{D_\beta^m} \left[\frac{\partial}{\partial t} w(t-s,x-y) - \frac{\partial}{\partial t} w(t,x)\right] \rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad + \int_{D_\beta^{m-1}} \left[\frac{\partial}{\partial t} w(t-s,x-y) - \frac{\partial}{\partial t} w(t,x) - y_m + y_{m+1}\right] \\ &\quad \times \rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y\Big| \end{aligned}$$

$$\leq \int\limits_{B_\beta(\mathbb{R}^{n+1})} \left|\frac{\partial}{\partial t}w(t-s,x-y)-\frac{\partial}{\partial t}w(t,x)\right| \rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

$$+ \int\limits_{B_\beta(\mathbb{R}^{n+1})} |y_{m+1}-y_m|\,\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

$$\leq \sup_{(s,y)\in B_\beta(\mathbb{R}^{n+1})} \left|\frac{\partial}{\partial t}w(t-s,x-y)-\frac{\partial}{\partial t}w(t,x)\right|$$

$$+ |y_{m+1}-y_m|$$

$$\leq M_w^t\sqrt{n+1}\beta+\eta_k, \tag{30}$$

where M_w^t is a Lipschitz constant of a function $\frac{\partial}{\partial t}w(\cdot,\cdot)$ of the class $C^1(T)$.

There exists real $0<\hat{\beta}^{k,i}<\hat{\beta}_1^k$ such that for every $0<\beta\leq\hat{\beta}^{k,i}$ we have

$$\left|\frac{\partial}{\partial t}w_2^{k,\beta}(t,x)-\frac{\partial}{\partial t}w_1^k(t,x)\right|<\frac{1}{i}\eta_k+\eta_k,\ (t,x)\in T, \tag{31}$$

(obviously, by taking $\hat{\beta}^{k,i}$, such that for every $0<\beta<\hat{\beta}^{k,i}$ we have $M_w^t\sqrt{n+1}\beta<\frac{1}{i}\eta_k$ or, more simply, $\hat{\beta}^{k,i}\leq\frac{1}{iM_w^t\sqrt{n+1}}\eta_k$ from (30) we see that the inequality (31) holds). ∎

Lemma 2. *For a given, fixed k and for every $i\in\mathbb{N}$ there exists $\check{\beta}^{k,i}>0$ such that for every $0<\beta\leq\check{\beta}^{k,i}$ and for all $(t,x)\in T\backslash Q$ the following inequality holds:*

$$\left|\frac{\partial}{\partial x}w_2^{k,\beta}(t,x)-\frac{\partial}{\partial x}w_1^k(t,x)\right|<\frac{1}{i}\eta_k. \tag{32}$$

Proof. We have the following estimate:

$$\left|\frac{\partial}{\partial x}w_2^{k,\beta}(t,x)-\frac{\partial}{\partial x}w_1^k(t,x)\right|$$

$$=\left|\int\limits_{B_\beta(\mathbb{R}^{n+1})}\frac{\partial}{\partial x}w(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y-\frac{\partial}{\partial x}w(t,x)\right|$$

$$=\int\limits_{B_\beta(\mathbb{R}^{n+1})}\left|\frac{\partial}{\partial x}w(t-s,x-y)-\frac{\partial}{\partial x}w(t,x)\right|\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

$$\leq\sup_{(s,y)\in B_\beta(\mathbb{R}^{n+1})}\left|\frac{\partial}{\partial x}w(t-s,x-y)-\frac{\partial}{\partial x}w(t,x)\right|$$

$$\leq M_w^x\sqrt{n+1}\beta, \tag{33}$$

where M_w^x is a Lipschitz constant of the function $\frac{\partial}{\partial x}w(\cdot,\cdot)$ of the class $C^1(T)$.

So there exists real $\check{\beta}^{k,i}>0$ such that for every $0<\beta\leq\check{\beta}^{k,i}$ we have

$$\left|\frac{\partial}{\partial x}w_2^{k,\beta}(t,x)-\frac{\partial}{\partial x}w_1^k(t,x)\right|<\frac{1}{i}\eta_k,\quad (t,x)\in T\backslash Q. \tag{34}$$

It is enough to choose $\check{\beta}^{k,i}$ such that for every $0<\beta<\check{\beta}^{k,i}$ we have $M_w^x\sqrt{n+1}\beta<\frac{1}{i}\eta_k$ or, more simply, $\check{\beta}^{k,i}\leq\frac{1}{iM_w^x\sqrt{n+1}}\eta_k$. ∎

Remark 1. For fixed k and for all $(t,x)\in T\backslash Q$ we have

$$\lim_{\beta\to 0}\frac{\partial}{\partial x}w_2^{k,\beta}(t,x)=\frac{\partial}{\partial x}w_1^k(t,x), \tag{35}$$

and the convergence is uniform.

In order to simplify the notation, we will define two auxiliary functions on $(T\backslash Q)\times U$, $\beta>0$:

$$\begin{aligned} g_1^k(t,x,u) &:= \frac{\partial}{\partial x}w_1^k(t,x)f(t,x,u)+L(t,x,u),\\ g_2^{k,\beta}(t,x,u) &:= \frac{\partial}{\partial x}w_2^{k,\beta}(t,x)f(t,x,u)+L(t,x,u). \end{aligned} \tag{36}$$

Lemma 3. *For fixed k and for all $(t,x,u)\in(T\backslash Q)\times U$ we have*

$$\lim_{\beta\to 0}g_2^{k,\beta}(t,x,u)=g_1^k(t,x,u), \tag{37}$$

and the convergence is uniform.

Proof. We must show that for every real $\epsilon>0$ there exists real $\tilde{\beta}$ such that for every $0<\beta\leq\tilde{\beta}$ and for all $(t,x,u)\in(T\backslash Q)\times U$ we have

$$\left|g_2^{k,\beta}(t,x,u)-g_1^k(t,x,u)\right|<\epsilon. \tag{38}$$

From Lemma 2 we have that for every $i\in\mathbb{N}$ there exists $\check{\beta}^{k,i}>0$ such that for every $0<\beta\leq\check{\beta}^{k,i}$ and for all $(t,x,u)\in T\times U$

$$\begin{aligned} &\left|g_2^{k,\beta}(t,x,u)-g_1^k(t,x,u)\right|\\ &\quad=\left|\frac{\partial}{\partial x}w_2^{k,\beta}(t,x)-\frac{\partial}{\partial x}w_1^k(t,x)\right||f(t,x,u)|\\ &\quad<\frac{1}{i}\eta_kM_f, \end{aligned} \tag{39}$$

where M_f is a constant limiting the function $|f(t,x,u)|$ from above on $(T\backslash Q)\times U$. Taking $\tilde{\beta}:=\check{\beta}^{k,i}$, where $i\in\mathbb{N}$ is such that $\frac{1}{i}\eta_kM_f\leq\epsilon$, we complete the proof. ∎

Let us introduce some additional symbols:

$$p_1^k(t,x) := \min_{u\in U} g_1^k(t,x,u) = g_1^k\big(t,x,u_1^k(t,x)\big),$$

$$\begin{aligned} p_2^{k,\beta}(t,x) &:= \min_{u\in U} g_2^{k,\beta}(t,x,u) \\ &= g_2^{k,\beta}\big(t,x,u_2^{k,\beta}(t,x)\big), \end{aligned} \tag{40}$$

where $u_1^k(t,x)$ and $u_2^{k,\beta}(t,x)$ are the values of control that minimize the respective functions g_1^k and $g_2^{k,\beta}$ at the point (t,x).

Lemma 4. *For fixed k and for all $(t,x)\in T\backslash Q$ we have*

$$\lim_{\beta\to 0} p_2^{k,\beta}(t,x) = p_1^k(t,x), \tag{41}$$

and this convergence is uniform.

Proof. In order to prove the uniform convergence, we have to show that for an arbitrarily chosen real number $\tilde{\epsilon} > 0$ there exists $\tilde{\delta} > 0$ such that for every $0 < \beta < \tilde{\delta}$, and for all $(t,x) \in T\backslash Q$ the following inequality holds:

$$\big|p_2^{k,\beta}(t,x) - p_1^k(t,x)\big| < \tilde{\epsilon}. \tag{42}$$

We partition the set $T\backslash Q$ into two sets: $Z' := \{(t,x) \in T\backslash Q : p_2^{k,\beta}(t,x) \geq p_1^k(t,x)\}$ and $Z'' := \{(t,x) \in T\backslash Q : p_2^{k,\beta}(t,x) < p_1^k(t,x)\}$. Obviously, we have $Z' \cup Z'' = T\backslash Q$ and $Z' \cap Z'' = \emptyset$.

Let $\tilde{\epsilon} > 0$ be some fixed real number and let $\tilde{\delta} > 0$ be a real number such that for every $0 < \beta < \tilde{\delta}$ and for all $(t,x,u) \in (T\backslash Q)\times U$:

$$\big|g_2^{k,\beta}(t,x,u) - g_1^k(t,x,u)\big| < \tilde{\epsilon}. \tag{43}$$

The existence of such a number $\tilde{\delta} > 0$ is guaranteed by Lemma 4.

Now we have two separate cases:

Case 1: $(t,x) \in Z'$. We have the following inequality:

$$\begin{aligned} 0 &\leq \big|p_2^{k,\beta}(t,x) - p_1^k(t,x)\big| = p_2^{k,\beta}(t,x) - p_1^k(t,x) \\ &= g_2^{k,\beta}\big(t,x,u_2^{k,\beta}(t,x)\big) - g_1^k\big(t,x,u_1^k(t,x)\big) \\ &\leq g_2^{k,\beta}\big(t,x,u_1^k(t,x)\big) - g_1^k\big(t,x,u_1^k(t,x)\big) \\ &\leq \big|g_2^{k,\beta}\big(t,x,u_1^k(t,x)\big) - g_1^k\big(t,x,u_1^k(t,x)\big)\big| < \tilde{\epsilon}. \end{aligned} \tag{44}$$

Case 2: $(t,x) \in Z''$. In this case we have following inequality:

$$\begin{aligned} 0 &\leq \big|p_2^{k,\beta}(t,x) - p_1^k(t,x)\big| = p_1^k(t,x) - p_2^{k,\beta}(t,x) \\ &= g_1^k\big(t,x,u_1^k(t,x)\big) - g_2^{k,\beta}\big(t,x,u_2^{k,\beta}(t,x)\big) \\ &\leq g_1^k\big(t,x,u_2^{k,\beta}(t,x)\big) - g_2^{k,\beta}\big(t,x,u_2^{k,\beta}(t,x)\big) \\ &\leq \big|g_2^{k,\beta}\big(t,x,u_2^{k,\beta}(t,x)\big) - g_1^k\big(t,x,u_2^{k,\beta}(t,x)\big)\big| < \tilde{\epsilon}. \end{aligned} \tag{45}$$

For all $(t,x) \in T\backslash Q$ one of these cases holds, which proves the theorem. ■

Remark 2. For fixed k and for any $i \in \mathbb{N}$ there exists real $\tilde{\beta}^{k,i} > 0$ such that for every $0 < \beta \leq \tilde{\beta}^{k,i}$, and for all $(t,x,u) \in (T\backslash Q)\times U$, we have

$$\big|p_2^{k,\beta}(t,x) - p_1^k(t,x)\big| < \frac{1}{i}\eta_k. \tag{46}$$

Proof. It is obvious because of the uniform convergence of $p_2^{k,\beta}(t,x)$ to $p_1^k(t,x)$ with respect to β on $(T\backslash Q)\times U$. ■

Some readers may have noticed that $\tilde{\beta}^{k,i}$ has not been computed effectively. We will give now a precise formula for $\tilde{\beta}^{k,i}$: in Lemma 3 we asserted that $\tilde{\beta}^{k,j} \leq \frac{1}{jM_w^x\sqrt{n+1}}\eta_k$. Because $\tilde{\epsilon} = \frac{1}{i}\eta_k$ we have from Lemma 4 that $\frac{1}{j}\eta_k M_f < \frac{1}{i}\eta_k$, which is equivalent to $iM_f < j$, and therefore $\tilde{\beta}^{k,i} = \tilde{\beta}^{k,iM_f} \leq \frac{1}{iM_f M_w^x\sqrt{n+1}}\eta_k$.

We are now ready to give the most important theorem of this paragraph.

Theorem 2. *For given, fixed k and for any $i \in \mathbb{N}$ there exists real $\bar{\beta}^{k,i} > 0$, such that for every $0 < \beta \leq \bar{\beta}^{k,i}$ and for all $(t,x) \in T\backslash Q$ the following inequality holds:*

$$\big|F_2^{k,\beta}(t,x) - F_1^k(t,x)\big| < \frac{2}{i}\eta_k + \eta_k. \tag{47}$$

Proof. It results immediately from the following estimate, where $0 < \beta \leq \bar{\beta}^{k,i} := \min(\hat{\beta}^{k,i}, \tilde{\beta}^{k,i})$:

$$\begin{aligned} &\big|F_2^{k,\beta}(t,x) - F_1^k(t,x)\big| \\ &= \left|\frac{\partial}{\partial t}w_2^{k,\beta}(t,x) + p_2^{k,\beta}(t,x) - \frac{\partial}{\partial t}w_1^k(t,x) - p_1^k(t,x)\right| \\ &\leq \left|\frac{\partial}{\partial t}w_2^{k,\beta}(t,x) - \frac{\partial}{\partial t}w_1^k(t,x)\right| + \big|p_2^{k,\beta}(t,x) - p_1^k(t,x)\big| \\ &< \frac{1}{i}\eta_k + \eta_k + \frac{1}{i}\eta_k = \frac{2}{i}\eta_k + \eta_k. \end{aligned} \tag{48}$$

■

As we can see, for every $(t,x) \in T\backslash Q$, all $i \in \mathbb{N}\backslash\{0,1\}$ and all $0 < \beta < \bar{\beta}^{k,i}$, the values of the function $F_2^{k,\beta}(\cdot,\cdot)$ can be estimated as follows:

$$-3\eta_k \leq -\frac{2}{i}\eta_k - \eta_k - \eta_k \leq F_2^{k,\beta}(t,x) \leq \frac{2}{i}\eta_k + \eta_k \leq 2\eta_k.$$

Obviously, the function $w_2^{k,\beta}(\cdot,\cdot)$ is of the class $C^1(T)$, yet it is not an ε-value function because $F_2^{k,\beta}(\cdot,\cdot)$ does not take non-positive values close to zero on T.

Of course, there is also one more important condition for our function to be an ε-value function, namely, the boundary condition. We will now give an estimate on how the values of $w_2^{k,\beta}(\cdot,\cdot)$ differ from the values of $w_1^k(\cdot,\cdot)$, which in turn are equal to the values of the original function $w(\cdot,\cdot)$.

Lemma 5. *For every given, fixed k and for every $i \in \mathbb{N}$ there exists $\mathring{\beta}^{k,i} > 0$ such that for every $0 < \beta \leq \mathring{\beta}^{k,i}$ and for all $(t,x) \in T \backslash Q$ the following inequality is satisfied:*

$$\left| w_2^{k,\beta}(t,x) - w_1^k(t,x) \right| < \Big(\frac{1}{i} + |b-t| \Big) \eta_k. \tag{49}$$

Proof. Take arbitrary $(t,x) \in T \backslash Q$. Then for some $m \in \{1,\dots,k\}$ the point $(t,x) \in P_m^k$. Since $F(\cdot,\cdot)$ is uniformly continuous on T, we can always find $\hat{\beta}_1^k$ such that for all $(s,y) \in B_{\hat{\beta}_1^k}(\mathbb{R}^{n+1})$ we have the estimate $|F(t-s,x-y) - F(t,x)| < \frac{1}{2}\eta_k$. Therefore, we have one of following cases, according to the location of (t,x) in the set P_m^k:

Case 1: $\quad \underset{(s,y)\in B_{\hat{\beta}_1^k}(\mathbb{R}^{n+1})}{\forall} \quad (t-s,x-y) \in P_m^k \cup P_{m-1}^k,$

Case 2: $\quad \underset{(s,y)\in B_{\hat{\beta}_1^k}(\mathbb{R}^{n+1})}{\forall} \quad (t-s,x-y) \in P_m^k \cup P_{m+1}^k.$

We will give here a proof only for Case 1, since the proof for Case 2 is analogous.

Let us introduce the following symbols: $D_\beta^m := \{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s,x-y) \in P_m^k\}$, $D_\beta^{m-1} := \{(s,y) \in B_\beta(\mathbb{R}^{n+1}) : (t-s,x-y) \in P_{m-1}^k\}$. Obviously, for every $0 < \beta < \hat{\beta}_1^k$ we have $B_\beta(\mathbb{R}^{n+1}) = D_\beta^m \cup D_\beta^{m-1}$, and $D_\beta^m \cap D_\beta^{m-1} = \emptyset$.

Thus the following inequality holds:

$$\begin{aligned}
&\left| w_2^{k,\beta}(t,x) - w_1^k(t,x) \right| \\
&= \Big| \int\limits_{B_\beta(\mathbb{R}^{n+1})} w_1^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y - w_1^k(t,x) \Big| \\
&= \Big| \int\limits_{D_\beta^m} w_{1,m}^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + \int\limits_{D_\beta^{m-1}} w_{1,m-1}^k(t-s,x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y - w_1^k(t,x) \Big| \\
&= \Big| \int\limits_{D_\beta^m} [w(t-s,x-y) - w(t,x)]\,\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + \int\limits_{D_\beta^{m-1}} \big[w(t-s,x-y) - w(t,x) \\
&\quad - y_m(b-t) + y_{m+1}(b-t) \big] \rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \Big| \\
&\leq \int\limits_{B_\beta(\mathbb{R}^{n+1})} |w(t-s,x-y) - w(t,x)|\,\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + \int\limits_{B_\beta(\mathbb{R}^{n+1})} |y_{m+1} - y_m|\,|b-t|\,\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\leq \sup_{(s,y)\in B_\beta(\mathbb{R}^{n+1})} |w(t-s,x-y) - w(t,x)| \\
&\quad + |y_{m+1} - y_m|\,|b-t| \\
&\leq M_w\sqrt{n+1}\beta + \eta_k\,|b-t|,
\end{aligned} \tag{50}$$

where M_w is a Lipschitz constant of a function $w(\cdot,\cdot)$ of the class $C^2(T)$.

There exists $0 < \mathring{\beta}^{k,i} < \hat{\beta}_1^k$ such that for every $0 < \beta \leq \mathring{\beta}^{k,i}$ we have

$$\left| w_2^{k,\beta}(t,x) - w_1^k(t,x) \right| < \frac{1}{i}\eta_k + \eta_k\,|b-t|,\ (t,x) \in T, \tag{51}$$

(obviously, by taking $\mathring{\beta}^{k,i} < \hat{\beta}_1^k$, such that for every $0 < \beta < \mathring{\beta}^{k,i}$ we have $M_w\sqrt{n+1}\beta < \frac{1}{i}\eta_k$ or, more simply, $\mathring{\beta}^{k,i} \leq \frac{1}{iM_w\sqrt{n+1}}\eta_k$, we see that the inequality (53) holds). ∎

Obviously, on the boundary (for $t = b$) we have the following estimate for all $i \in \mathbb{N}$ and for every $0 < \beta \leq \mathring{\beta}^{k,i}$

$$\left| w_2^{k,\beta}(b,x) - w_1^k(b,x) \right| < \eta_k. \tag{52}$$

We will now construct a function $(t,x) \to w_3^{k,\beta}(t,x)$ that will be a function of the class $C^1(T)$, and the respective function $F_3^{k,\beta}(\cdot,\cdot)$ will take non-positive values close to zero almost everywhere on T. Let us introduce the following definitions:

$$w_3^{k,\beta}(t,x) := w_2^{k,\beta}(t,x) + 3\eta_k(b-t),\ (t,x) \in T, \tag{53}$$

$$\begin{aligned}
F_3^{k,\beta}(t,x) &:= \frac{\partial}{\partial t} w_3^{k,\beta}(t,x) \\
&\quad + \min_{u\in U} \Big\{ \frac{\partial w_3^{k,\beta}}{\partial x}(t,x) f(t,x,u) \\
&\quad + L(t,x,u) \Big\}, \qquad (t,x) \in T.
\end{aligned} \tag{54}$$

Obviously, the function $w_3^{k,\beta}(\cdot,\cdot)$ is of the class $C^1(T)$. At the same time, by simple calculations we have that for all $0 < \beta < \bar{\beta}^{k,i}$:

$$F_3^{k,\beta}(t,x) = F_2^{k,\beta}(t,x) - 3\eta_k, \tag{55}$$

so the following estimation holds:

$$-6\eta_k \leq F_3^{k,\beta}(t,x) \leq -\eta_k. \tag{56}$$

In addition to that we have the following inequality for the boundary condition, which holds for all $0 < \beta < \mathring{\beta}^{k,i}$:

$$l(x) + \frac{\varepsilon}{2} - \eta_k \leq w_3^{k,\beta}(b,x) \leq l(x) + \frac{\varepsilon}{2} + \eta_k,$$

which in turn means that for all $k > k_\varepsilon$, where k_ε is such that it also holds $6\eta_{k_\varepsilon} \leq \frac{\varepsilon}{2}$. Therefore, the function $w_3^{k,\beta}(\cdot,\cdot)$ satisfies (9) for all $0 < \beta < \min(\bar{\beta}^{k,i}, \mathring{\beta}^{k,i})$, and is a candidate for an ε-value function according to Theorem 1.

By simple calculations we obtain that $k_\varepsilon = 12|\kappa_g - \kappa_d|/\varepsilon$. The values of $\bar{\beta}^{k,i}$ and $\mathring{\beta}^{k,i}$ are easy to calculate from the corresponding Lipschitz constants.

3.2. Algorithm for Evaluating ε-Value Functions

The algorithm for calculating the ε-value function is quite straightforward. However, it is a numerical algorithm, and not a computer algorithm. We shall discuss improvements needed for converting it into a computer program. Such an implementation is in fact being written as part of our research and will soon be submitted for publication. The main aim of this section is to provide a kind of summary for the method presented above, rather than a computer-oriented algorithm. Therefore we require the user to submit data, which in a fully developed algorithm would be calculated by the program. Likewise, the algorithm offers no assistance in establishing the set T.

Input: The algorithm requires the following values to be calculated by the user: all required Lipschitz constants i.e., M_w, M_w^t, M_w^x and M_f, the values of lower and upper limits of F : κ_d and κ_g, the value of ε, the point (t,x) for which the ε-value function should be calculated and, of course, the function $w(\cdot,\cdot)$ of the class $C^2(T)$ satisfying the boundary condition. Without the boundary condition being satisfied the results will usually be wrong. The aforementioned values have to be provided by the user. However, in a computer implementation we will calculate suitable approximations.

Output: The output of the algorithm is the value of the ε-value function at a given point (t,x). The algorithm calculates the value of an ε-value function at a single point provided by the user.

The algorithm consists of the following steps:

Step 1: Calculate the values $k_\varepsilon := 12|\kappa_g - \kappa_d|/\varepsilon$ and $\eta_{k_\varepsilon} := \varepsilon/12$, and for all following steps set $k = k_\varepsilon$ and $\eta_k = \eta_{k_\varepsilon}$.

Step 2: Compute the value $\beta_{\min}$, such that for all $0 < \beta < \beta_{\min}$ the function $w_3^{k,\beta}(\cdot,\cdot)$ is an ε-value function, $K := \frac{1}{(\int_{B_\beta(n+1)} \frac{1}{\beta^2} e^{-\beta^2/(\beta^2-(t^2+x^2))} dx)^{-1}}$ based on the respective Lipschitz constants as follows:

$$\beta_{\min} = \min(\bar{\beta}^{k,i}, \mathring{\beta}^{k,i}) = \min\left\{\hat{\beta}^{k,i}, \tilde{\beta}^{k,i}, \mathring{\beta}^{k,i}\right\},$$

where we assume $i = 1$, so that

$$\beta_{\min} = \min\left\{\frac{1}{M_w^t\sqrt{n+1}}\eta_k, \frac{1}{M_f M_w^x\sqrt{n+1}}\eta_k, \frac{1}{M_w\sqrt{n+1}}\eta_k\right\}.$$

Step 3: For $\beta = \beta_{\min}$ define the function $\rho_\beta(t,x)$ as follows:

$$\rho_\beta(t,x) := \begin{cases} K\frac{1}{\beta^2}e^{-\beta^2/\left(\beta^2-\left(t^2+x^2\right)\right)} & \text{for } \sqrt{t^2+x^2} \leq \beta, \\ 0 & \text{for } \sqrt{t^2+x^2} > \beta. \end{cases}$$

Determine a constant K such that we shall have $\int_{\mathbb{R}^{n+1}} \rho_\beta(t,x)\,\mathrm{d}t\,\mathrm{d}x = 1$. In fact, we simply set:

Step 4: Determine

$$\begin{aligned} j &= \left\lfloor \frac{1}{\eta_k}|F(t,x) - \kappa_d| \right\rfloor + 1 \\ &= \left\lfloor \frac{1}{\eta_k}\Big|\frac{\partial}{\partial t}w(t,x) \right. \\ &\qquad \left. + \min_{u\in U}\left\{\frac{\partial w}{\partial x}(t,x)f(t,x,u) + L(t,x,u)\right\} - \kappa_d\Big| \right\rfloor + 1, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the integer part of x, i.e., the greatest integer number that is less than x. (We identify the right $j \in \{1,\ldots,k\}$ for whose j we have $(t,x) \in P_j$, where P_j are defined in accordance with (20) and (21).)

Step 5: Compute

$$w_{1,j}^k(t,x) = w(t,x) + y_{j+1}(b-t)$$

according to (22).

Step 6: Compute

$$w_2^{k,\beta}(t,x) = \int\limits_{B_\beta(\mathbb{R}^{n+1})} w_1^k(t-s, x-y)\rho_\beta(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

according to (27).

Step 7: Determine

$$w_3^{k,\beta}(t,x) = w_2^{k,\beta}(t,x) + 3\eta_k(b-t)$$

according to (55).

The value calculated in Step 7 is the output of the algorithm.

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