HYBRID STABILIZATION OF DISCRETE–TIME LTI SYSTEMS WITH TWO QUANTIZED SIGNALS

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We consider stabilizing a discrete-time LTI (linear time-invariant) system via state feedback where both the quantized state and control input signals are involved. The system under consideration is stabilizable and stabilizing state feedback has been designed without considering quantization, but the system's stability is not guaranteed due to the quantization effect. For this reason, we propose a hybrid quantized state feedback strategy asymptotically stabilizing the system, where the values of the quantizer parameters are updated at discrete time instants. We also extend the result to the case of static output feedback.

Keywords: discrete-time LTI system, hybrid stabilization strategy, quantizer, state feedback, output feedback

1. Introduction

In classical feedback control theory, various signals or data in the control loop have been assumed to be passed directly without information loss, except in saturated systems. However, this is not true in many real applications. For example, in networked control systems (Bushnell, 2001), where all signals are transferred through a network, package dropouts or data transfer rate limitations always happen. Another important aspect, which is well known in the signal processing area, is signal quantization. Since quantization always exists in computer-based control systems, many researchers have begun to consider the analysis and design problems for control systems involving various quantization methods. Delchamps (1990) addressed the problem of stabilizing an unstable linear system by means of quantized state feedback, i.e., the state feedback where the measurements of the system state are quantized. The quantizer in (Delchamps, 1990) takes values in a countable set. Brockett and Liberzon (2000) defined a quantizer taking values in a finite set and considered quantized feedback stabilization for linear systems. It was shown that if it is possible to change the sensitivity of the quantizer on the basis of the available quantized measurements, then a hybrid control strategy, for both continuous- and discrete-time systems, can be designed to guarantee global asymptotic stability. Noting that the approach in (Brockett and Liberzon, 2000) exploits the possibility of making discrete online adjustments of quantizer parameters, Liberzon (2003) extended the approach to more general nonlinear systems with general types of quantizers affecting the system state, the measured output, or the control input.

Motivated by the above works, we consider the stabilization problem for a discrete-time LTI system via state feedback involving both quantized states and control inputs, see Fig. 1 for the structure of the closed-loop feedback system. The system under consideration is supposed to be stabilizable and stabilizing state feedback has been designed without taking the quantization into account. However, the system's states are quantized by the first quantizer before they are passed on to the controller, and the control inputs are quantized by the second quantizer before they are passed on to the system. This is a

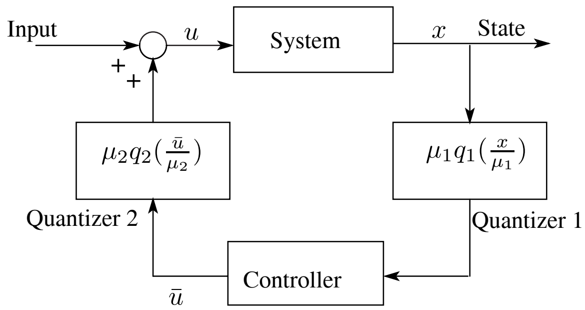

Fig. 1. Feedback Control Systems with Quantized States and Control Inputs.

natural setting in networked control systems, where all information (reference inputs, plant outputs, control inputs, etc.) is exchanged through a network among control system components (sensors, controller, actuators, etc.). Due to the quantization effects, the desired degree of system stability cannot be guaranteed, and in the worst case the system may become unstable. For example, if a discrete-time system is stabilized by the constant feedback $u(k) = 1.995$ with a very small decay rate due to some constraints, then the very simple "round-off" quantizer on the control input results in $u(k) = 2.000$, which will certainly affect the system's stability (a detailed discussion of the analysis and design of the generalized "round-off" quantizer can be found in Brockett and Liberzon, 2000). To overcome such a difficulty, we adopt the two quantizers in a general form as in (Liberzon, 2003), and then propose a hybrid quantized state feedback strategy where the values of the quantizer parameters are updated at discrete time instants. It should be noted that the results and proofs in this paper are not a trivial extension of those in (Liberzon, 2003), where hybrid quantized feedback stabilization was dealt with for continuous-time systems with a single quantizer.

The rest of this paper is organized as follows: Section 2 gives the definition and the property of the generalized quantizer, and Section 3 describes the control problem at hand. Section 4 presents a result concerning invariant regions for the closed-loop system when the quantizers' parameters are fixed. Section 5 then establishes a hybrid stabilization strategy, and Section 6 is an illustrative example. Section 7 extends the consideration to the case of output feedback, and, finally, Section 8 gives some concluding remarks.

2. Quantizer

First, we give the definition of a quantizer in a general form as in (Liberzon, 2003). Let $z \in \mathbb{R}^l$ be the variable being quantized. A *quantizer* is defined as a piecewise constant function $q : \mathbb{R}^l \to \mathcal{D}$, where $\mathcal{D}$ is a finite subset of $\mathbb{R}^l$. This leads to a partition of $\mathbb{R}^l$ into a finite number of quantization regions of the form $\{z \in \mathbb{R}^l : q(z) = i\}$, $i \in \mathcal{D}$. These quantization regions are not assumed to have any particular shapes. We assume that there exist positive real numbers M_0 and Δ_0 such that the following conditions hold: (1) if $|z| \le M_0$, then $|q(z) - z| \le \Delta_0$; (2) if $|z| > M_0$, then $|q(z)| > M_0 - \Delta_0$. Throughout this paper, we denote by $|\cdot|$ the standard Euclidean norm in the n-dimensional vector space $\mathbb{R}^n$, and let $\|\cdot\|$ stand for the corresponding induced matrix norm in $\mathbb{R}^{n\times n}$. Condition (1) gives a bound on the quantization error when the quantizer does not saturate. Condition (2) provides a way to detect the possibility of saturation. We will refer to M_0 and Δ_0 as the *range of* q and the *quantization error*, respectively. We also assume that $q(x) = 0$ for x in some neighborhood of the origin. An example satisfying the above requirements is the quantizer with rectangular quantization regions given in (Brockett and Liberzon, 2000; Liberzon, 2000).

In the control strategy to be developed below, we will use quantized measurements of the form

$$q_\mu(z) \triangleq \mu q\Big(\frac{z}{\mu}\Big), \tag{1}$$

where $\mu > 0$ is the quantizer parameter. The range of this quantizer is $M_0\mu$ and the quantization error is $\Delta_0\mu$. We can view μ as a "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with a larger range and a larger quantization error, while decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error.

As can be seen later, we will consider two general quantizers in the feedback control system, one dealing with the system state (or the measured output) and the other dealing with the control inputs. As was also pointed out in (Liberzon, 2003), the quantizers can be viewed as devices which convert a real-valued signal into a piecewise constant one taking values in a finite set. We will propose updating the value of the quantizer parameters at discrete time instants, depending only on time, and thus they can be regarded as another discrete state of the resultant closed-loop system, which may be implemented independently inside the quantizers. In this sense, we view the whole system (depicted in Fig. 1) as a hybrid dynamical system.

3. Problem Description

Consider the discrete-time LTI system

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, and A, B are constant matrices of appropriate dimensions. Throughout this paper, we assume that the system (2) is stabilizable via state feedback, and the stabilizing state feedback $u(k) = Kx(k)$ has been designed so that $A + BK$ is Schur stable. Then, by the standard Lyapunov stability theory, there exist positive definite matrices P and Q such that

$$(A + BK)^T P(A + BK) - P = -Q. \tag{3}$$

We will let $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. Since P is positive definite, the inequality

$$\lambda_m(P)\,|x(k)|^2 \le x^T(k)Px(k) \le \lambda_M(P)\,|x(k)|^2 \tag{4}$$

holds for any $x(k)$.

As is explained above, we deal with the situation where only quantized measurements of the state $x(k)$ are available in the controller and, furthermore, only quantized data of the input $u(k)$ are available in the system. Assume that the two quantizers q_1 and q_2 are characterized by quantization ranges M_1, M_2 and quantization errors Δ_1, Δ_2, respectively, and for the two quantizers we use parameters (“zoom” variables) μ_1 and μ_2, respectively. Therefore, the input to the controller is

$$\bar{x}(k) = \mu_1 q_1\left(\frac{x(k)}{\mu_1}\right), \tag{5}$$

and thus the input to the system is

$$\begin{aligned} u(k) &= \mu_2 q_2\left(\frac{\bar{u}(k)}{\mu_2}\right) = \mu_2 q_2\left(\frac{K\bar{x}(k)}{\mu_2}\right) \\ &= \mu_2 q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right). \end{aligned} \tag{6}$$

Then, for any fixed positive scalars μ_1 and μ_2, the closed-loop system composed of the system (2) and the controller (6) is given by

$$\begin{aligned} x(k+1) &= Ax(k) + B\mu_2 q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right) \\ &= (A + BK)x(k) - Bd(k), \end{aligned} \tag{7}$$

where

$$d(k) = \mu_2\left(\frac{Kx(k)}{\mu_2} - q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right)\right). \tag{8}$$

Now, the control problem is very natural. When the above closed-loop system is not asymptotically stable for any fixed positive scalars μ_1 and μ_2, we wish to adjust them appropriately on-line so that the entire system is asymptotically stable.

4. Invariant Regions

In this section, we characterize the behavior of the trajectories of the system (7) with fixed μ_1 and μ_2 in the following result.

Lemma 1. *Fix an arbitrary scalar $\varepsilon \in (0,1)$ and assume that*

$$\sqrt{\lambda_m(P)}M > \sqrt{\lambda_M(P)}\Theta\|K\|\Delta\frac{1}{1-\varepsilon}, \tag{9}$$

where

$$\begin{aligned} M &= M_2 - \frac{\|K\|\mu_1}{\mu_2}(\Delta_1 + M_1), \\ \Delta &= \frac{\|K\|\mu_1}{\mu_2}\Delta_1 + \Delta_2, \\ \Theta &= \frac{\alpha + \sqrt{\alpha^2 + \beta\lambda_m(Q)(1-\varepsilon)}}{\lambda_m(Q)}, \\ \alpha &= \|(A+BK)^T PB\|, \quad \beta = \|B^T PB\|. \end{aligned} \tag{10}$$

Then, the ellipsoids

$$\mathcal{R}_1(\mu_1,\mu_2) \triangleq \left\{x(k) : x^T(k)Px(k) \le \frac{\lambda_m(P)M^2{\mu_2}^2}{\|K\|^2}\right\} \tag{11}$$

and

$$\mathcal{R}_2(\mu_1,\mu_2) \triangleq \left\{x(k) : x^T(k)Px(k) \le \frac{\lambda_M(P)\Theta^2\Delta^2{\mu_2}^2}{(1-\varepsilon)^2}\right\} \tag{12}$$

are invariant regions for the system (7). Moreover, all solutions of (7) that start in the ellipsoid $\mathcal{R}_1(\mu_1,\mu_2)$ enter the smaller ellipsoid $\mathcal{R}_2(\mu_1,\mu_2)$ in a finite number of steps.

Proof. When $|x(k)/\mu_1| \le M_1$, it is easy to see that

$$\left|q_1\left(\frac{x(k)}{\mu_1}\right)\right| - \left|\frac{x(k)}{\mu_1}\right| \le \left|q_1\left(\frac{x(k)}{\mu_1}\right) - \frac{x(k)}{\mu_1}\right| \le \Delta_1, \tag{13}$$

and thus

$$\left|q_1\left(\frac{x(k)}{\mu_1}\right)\right| \le \Delta_1 + \left|\frac{x(k)}{\mu_1}\right| \le \Delta_1 + M_1. \tag{14}$$

Furthermore, using the definition of M, we obtain

$$\begin{aligned} \left|\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right| &\le \frac{\|K\|\mu_1}{\mu_2}\left|q_1\left(\frac{x(k)}{\mu_1}\right)\right| \\ &\le \frac{\|K\|\mu_1}{\mu_2}(\Delta_1 + M_1) \\ &= M_2 - M < M_2, \end{aligned} \tag{15}$$

($M > 0$ is guaranteed by (9)), and thus

$$\left| q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right) - \frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right| \le \Delta_2, \quad (16)$$

according to the quantizer property. Then, using the definition of Δ, we get

$$\begin{aligned} &\left|\frac{Kx(k)}{\mu_2} - q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right)\right| \\ &\quad \le \left|\frac{K\mu_1}{\mu_2}\left(\frac{x(k)}{\mu_1} - q_1(\frac{x(k)}{\mu_1})\right)\right| \\ &\qquad + \left|q_2\left(\frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right) - \frac{K\mu_1 q_1(\frac{x(k)}{\mu_1})}{\mu_2}\right| \\ &\quad \le \frac{\|K\|\mu_1}{\mu_2}\Delta_1 + \Delta_2 = \Delta. \end{aligned} \quad (17)$$

Therefore, whenever $|x(k)/\mu_1| \le M_1$, the increment in the Lyapunov function candidate $V(k) = x^T(k)Px(k)$ along the solutions of the closed-loop system (7) can be computed as

$$\begin{aligned} &V(k+1) - V(k) \\ &\quad = x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &\quad = -x^T(k)Qx(k) - 2x^T(k)(A+BK)^T PBd(k) \\ &\qquad + d^T(k)B^T PBd(k) \\ &\quad \le -\lambda_m(Q)\,|x(k)|^2 + 2\,|x(k)|\,\alpha\Delta\mu_2 \\ &\qquad + \beta\Delta^2{\mu_2}^2, \end{aligned} \quad (18)$$

where the Lyapunov equation (3) is used.

Furthermore, it is easy to deduce that whenever

$$|x(k)| > \frac{\Theta\Delta\mu_2}{1-\varepsilon} \quad (19)$$

holds for a positive scalar ϵ, we obtain

$$\begin{aligned} V(k+1) - V(k) &\le -\lambda_m(Q)\,|x(k)|^2 + 2\,|x(k)|\,\alpha\Delta\mu_2 \\ &\quad + \beta\Delta^2{\mu_2}^2 \\ &< -\varepsilon\lambda_m(Q)\,|x(k)|^2, \end{aligned} \quad (20)$$

which means that the Lyapunov function candidate decreases exponentially in the region defined by (19).

Define the balls

$$\mathcal{B}_1(\mu_1,\mu_2) \triangleq \left\{x(k) : |x(k)| \le \frac{M\mu_2}{\|K\|}\right\} \quad (21)$$

and

$$\mathcal{B}_2(\mu_1,\mu_2) \triangleq \left\{x(k) : |x(k)| \le \frac{\Theta\Delta\mu_2}{1-\varepsilon}\right\}. \quad (22)$$

In view of the assumption (9), we can easily confirm the relation

$$\begin{aligned} &\mathcal{B}_2(\mu_1,\mu_2) \subset \mathcal{R}_2(\mu_1,\mu_2) \\ &\qquad \subset \mathcal{R}_1(\mu_1,\mu_2) \subset \mathcal{B}_1(\mu_1,\mu_2), \end{aligned} \quad (23)$$

which is described in Fig. 2.

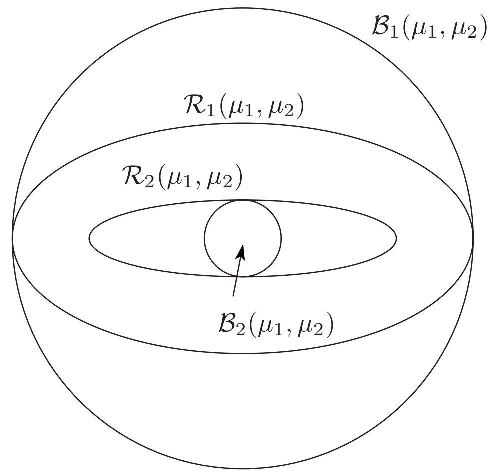

Fig. 2. The inclusion relation of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{R}_1, \mathcal{R}_2$.

Using the fact that $V(k)$ decreases for any $x(k)$ in the exterior of $\mathcal{B}_2(\mu_1,\mu_2)$, we immediately see that the ellipsoids $\mathcal{R}_1(\mu_1,\mu_2)$ and $\mathcal{R}_2(\mu_1,\mu_2)$ are both invariant regions for the system (7).

Now, we use the inequality (20) concerning the increment in x^TPx to show that the trajectories starting in $\mathcal{R}_1(\mu_1,\mu_2)$ reach $\mathcal{R}_2(\mu_1,\mu_2)$ in a finite number of steps.

Suppose that at some time instant k_0, $x(k_0) \in \mathcal{R}_1(\mu_1,\mu_2)$, but $x(k_0) \notin \mathcal{R}_2(\mu_1,\mu_2)$. According to the definition of $\mathcal{R}_1(\mu_1,\mu_2)$, we have

$$V(k_0) = x^T(k_0)Px(k_0) \le \frac{\lambda_m(P)M^2{\mu_2}^2}{\|K\|^2}. \quad (24)$$

Then, using (20), we obtain

$$\begin{aligned} &V(k_0+1) - V(k_0) < -\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}V(k_0) \\ &\qquad \Longleftrightarrow V(k_0+1) < \left(1 - \varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)V(k_0). \end{aligned} \quad (25)$$

By induction, for any $S > 1$,

$$V(k_0+S) < \frac{\lambda_m(P)M^2{\mu_2}^2}{\|K\|^2}\left(1 - \varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^S. \quad (26)$$

Therefore, if S is large enough so that

$$\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^S \le \frac{\lambda_M(P)\Theta^2\|K\|^2\Delta^2}{\lambda_m(P)M^2(1-\varepsilon)^2}, \tag{27}$$

then we obtain

$$\begin{aligned} V(k_0+S) &< \frac{\lambda_m(P)M^2{\mu_2}^2}{\|K\|^2}\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^S \\ &\le \frac{\lambda_M(P)\Theta^2\Delta^2{\mu_2}^2}{(1-\varepsilon)^2}, \end{aligned} \tag{28}$$

and thus $x(k_0+S) \in \mathcal{R}_2(\mu_1,\mu_2)$. Noting that S is an integer denoting the number of steps from the time instant k_0, we obtain the following estimate of the number of steps for moving from $\mathcal{R}_1(\mu_1,\mu_2)$ to $\mathcal{R}_2(\mu_1,\mu_2)$ as

$$\begin{aligned} S_0 = \Bigg\lceil &\frac{\log(\lambda_M(P)\Theta^2\|K\|^2\Delta^2)}{\log\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)} \\ &- \frac{\log(\lambda_m(P)M^2(1-\varepsilon)^2)}{\log\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)}\Bigg\rceil \end{aligned} \tag{29}$$

by using (27). Here, $\lceil r\rceil$ denotes the minimum integer greater than r. This completes the proof. ■

Remark 1. By analogy with continuous-time systems, we used the same name "invariant region" in Lemma 1. However, special attention is needed to deal with the special case where some state $x(k)$ in the region $\mathcal{B}_2$ jumps out of $\mathcal{R}_2$ at $x(k+1)$. Since such $x(k+1)$ goes back to $\mathcal{R}_2$ quickly according to (20) (which means that $V(x)$ decreases outside $\mathcal{B}_2$), we did not make a clear distinction between this case and other cases precisely in the proof and regarded it as a discrete-time version of the invariant region. Simulation examples also show that this treatment will not affect the whole system stability.

5. Hybrid Stabilization Strategy

In this section, we describe constructively our hybrid state feedback strategy, where the quantizer parameters μ_1 and μ_2 are updated at discrete time instants. As will be seen later, an open-loop "zooming-out" stage is followed by a closed-loop "zooming-in" stage.

We consider a strategy which always satisfies

$$\mu_1 = \zeta\mu_2, \tag{30}$$

where ζ is a positive constant. Although in the present discussion ζ can be arbitrary, it should be adjusted in real applications. According to (10) and (30), we obtain

$$\begin{aligned} M &= M_2 - \|K\|\zeta(\Delta_1+M_1), \\ \Delta &= \|K\|\zeta\Delta_1+\Delta_2. \end{aligned} \tag{31}$$

Now, we state and prove the first main result in this paper.

Theorem 1. *If M is large enough compared with Δ in (31) so that*

$$\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}}M > \frac{\alpha+\sqrt{\alpha^2+\beta\lambda_m(Q)}}{\lambda_m(Q)}\|K\|\Delta \tag{32}$$

holds with some positive scalar ζ, then there exists a hybrid quantized state feedback strategy assuring that the system (7) is globally asymptotically stable.

Proof. *"Zooming-out" stage*. Set the control input u equal to 0. Let $\mu_1(0) = \zeta\mu_2(0) = 1$. Then increase $\mu_1(k)$ (and thus $\mu_2(k)$) in a "piecewise" constant manner (when viewed in a continuous-time domain), fast enough to dominate the rate of growth of $\|A^k\|$. For example, one can fix a positive integer $\kappa > 1$ and let

$$\begin{aligned} &\mu_1(k) = \zeta\mu_2(k) = 1 && \text{when} \quad k\in[0,\kappa), \\ &\mu_1(k) = \zeta\mu_2(k) = \kappa\|A\|^{2\kappa} && \text{when} \quad k\in[\kappa,2\kappa), \\ &\mu_1(k) = \zeta\mu_2(k) = 2\kappa\|A\|^{4\kappa} && \text{when} \quad k\in[2\kappa,3\kappa), \end{aligned} \tag{33}$$

and so on.

Let us observe the norm change in $x(k)/\mu_1(k)$. When $k \in [\kappa, 2\kappa)$, from $|x(k)| \le \|A\|^k|x_0|$, $\mu_1(k) = \kappa\|A\|^{2\kappa}$ we obtain

$$\frac{|x(k)|}{\mu_1(k)} \le \frac{|x_0|}{\kappa\|A\|^{2\kappa-k}} \le \frac{|x_0|}{\kappa}. \tag{34}$$

When $k \in [2\kappa, 3\kappa)$, $\mu_1(k) = 2\kappa\|A\|^{4\kappa}$,

$$\frac{|x(k)|}{\mu_1(k)} \le \frac{|x_0|}{2\kappa\|A\|^{4\kappa-k}} \le \frac{|x_0|}{2\kappa\|A\|^{\kappa}}. \tag{35}$$

Thus, we can find an integer $\hat{k}$ such that, for all $k \ge \hat{k}$,

$$\left|\frac{x(k)}{\mu_1(k)}\right| \le M_1. \tag{36}$$

Next, we fix $\mu_1(k)$ as $\mu_1(\hat{k})$ and increase $\mu_2(k)$ until

$$\frac{|x(k)|}{\mu_2(k)M_2 - \|K\|\mu_1(k)(\Delta_1+M_1)} \le \sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}}\frac{1}{\|K\|}$$

is satisfied for some k_0. This is possible since

$$M = M_2 - \|K\|\zeta(\Delta_1+M_1) > 0 \tag{37}$$

according to (32). Then, it is easy to see that

$$|x(k_0)| \le \sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}} \frac{M\mu_2(k_0)}{\|K\|}, \tag{38}$$

which means that $x(k_0)$ belongs to the ellipsoid $\mathcal{R}_1(\mu_1, \mu_2)$ with $\mu_1 = \mu_1(k_0)$, $\mu_2 = \mu_2(k_0)$.

"Zooming-in" stage. For $k \ge k_0$ we use the control law (6). In view of (32), we can always choose a scalar $0 < \varepsilon < 1$ such that the inequality

$$\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}} M > \frac{\alpha + \sqrt{\alpha^2 + \beta\lambda_m(Q)(1-\varepsilon)}}{\lambda_m(Q)} \|K\|\Delta \frac{1}{1-\varepsilon} \tag{39}$$

holds. Noting that this inequality is equivalent to (9), we can apply Lemma 1 right now.

We know that $x(k_0)$ belongs to $\mathcal{R}_1(\mu_1, \mu_2)$ with $(\mu_1, \mu_2) = (\mu_1(k_0), \mu_2(k_0))$. Since $\mu_1(k)$ and $\mu_2(k)$ are proportional to each other, we only describe how to change the value of $\mu_2(k)$. Let $\mu_2(k) = \mu_2(k_0)$ for $k \in [k_0, k_0 + S_0)$, where S_0 is given by (29). Then, according to Lemma 1, $x(k_0 + S_0)$ belongs to the ellipsoid $\mathcal{R}_2(\mu_1, \mu_2)$ given by (12) with $(\mu_1, \mu_2) = (\mu_1(k_0), \mu_2(k_0))$. For $k \in [k_0 + S_0, k_0 + 2S_0)$, let

$$\mu_2(k) = \Omega\mu_2(k_0), \tag{40}$$

where

$$\Omega := \frac{\sqrt{\lambda_M(P)}\Theta\Delta\|K\|}{\sqrt{\lambda_m(P)}M(1-\varepsilon)}. \tag{41}$$

We have $\Omega < 1$ by (9) or (39), and hence $\mu_2(k_0 + S_0) < \mu_2(k_0)$. The ellipsoid $\mathcal{R}_2(\mu_1, \mu_2)$ with the old value $\mu_2 = \mu_2(k_0)$ is the same as the ellipsoid $\mathcal{R}_1(\mu_1, \mu_2)$ with the new value $\mu_2 = \mu_2(k_0 + S_0)$. This means that we can continue the analysis for $k > k_0 + S_0$ as before. Namely, $x(k_0 + 2S_0)$ belongs to the ellipsoid $\mathcal{R}_2(\mu_1, \mu_2)$ defined by (12) with $\mu_2 = \mu_2(k_0 + S_0)$. For $k \in [k_0 + 2S_0, k_0 + 3S_0)$, let $\mu_2(k) = \Omega\mu_2(k_0 + S_0) = \Omega^2\mu_2(k_0)$. Repeating this procedure, we obtain the desired control strategy. In fact, since $\Omega < 1$, we have $\mu_2(k) \to 0$, and thus $\mu_1(k) \to 0$ as $k \to \infty$. In view of the definitions of $\mathcal{R}_1(\mu_1, \mu_2)$ and $\mathcal{R}_2(\mu_1, \mu_2)$, we see that $x(k) \to 0$ as $k \to \infty$. The detailed proof of the asymptotic stability of the equilibrium $x = 0$ of the system in the Lyapunov sense is similar to the final part of the proof of Theorem 1 in (Liberzon, 2003), and thus it is omitted. ■

Remark 2. In the proof of Theorem 1, we updated the values of μ_2 at the time instants k_0, $k_0 + S_0$, $k_0 + 2S_0$, $\cdots$. Since the ellipsoids in the proof are invariant regions for the closed-loop system, we can also choose the time instants $k_1, k_2, \ldots$ of updating μ_2 satisfying $k_i - k_{i-1} \ge S_0$, $i \ge 1$. The constant S_0 is usually referred to as a lower bound of *dwell time* (Hespanha and Morse, 1999; Zhai *et al.*, 2001) in the analysis and design of hybrid and switched systems.

Remark 3. As is shown in the proof of Theorem 1, the switching strategy (determining the time instants of updating μ_1 and μ_2) is *time-based*, in the sense that the values of the quantizer parameters are updated at pre-computed time at which the system state is guaranteed to enter a certain region. Alternatively, an *event-based* switching strategy, i.e., using the quantized measurements to determine when x enters the desired region, can be employed for the same purpose, cf. (Liberzon, 2000)

6. Simulation

Consider the system (2) for

$$A = \begin{bmatrix} 1.11 & 0.60 \\ 0.50 & -1.31 \end{bmatrix}, \quad B = \begin{bmatrix} 0.50 \\ -0.70 \end{bmatrix}, \tag{42}$$

together with the pre-designed stabilizing state feedback gain

$$K = [\,-2.00 \quad -1.00\,]\,. \tag{43}$$

We set

$$Q = \begin{bmatrix} 3.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \tag{44}$$

in the Lyapunov equation (3) to obtain

$$P = \begin{bmatrix} 3.2552 & -0.8566 \\ -0.8566 & 23.4705 \end{bmatrix}. \tag{45}$$

Then we use

$$\varepsilon = 0.50, \quad \kappa = 5.00 \tag{46}$$

in our hybrid stabilization strategy, and find that $\mu_1(k)$ is fixed when $k = 34$, the "zooming-out" stage is finished when $k_0 = 124$. After that, repeating the procedure in the "zooming-in" stage, we see that the norm of the system state $|x(k)|$ varies between $\mathcal{B}_1(k) = M^2\mu_2(k)^2/\|K\|^2$ and $\mathcal{B}_2(k) = \Theta^2\Delta^2\mu_2(k)^2/(1-\varepsilon)^2$, and converges to zero quickly, as depicted in Fig. 3 ($x(0) = [4.0 \;\; 6.0]^T$).

7. Extension to Output Feedback

In this section, we note that since the updating method of the quantizer parameters is time-based, the approach can be easily extended to the case of static/dynamic output feedback. Fig. 4 depicts a feedback control system with quantized measured outputs and control inputs. Here,

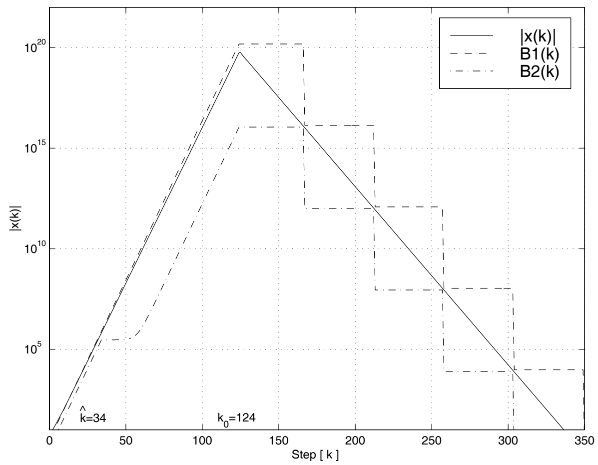

Fig. 3. Evolution of $\mathcal{B}_1(k)$, $\mathcal{B}_2(k)$, and $|x(k)|$.

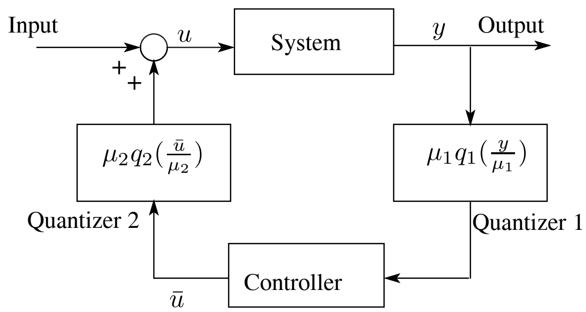

Fig. 4. Feedback control system with quantized measured outputs and control inputs.

with no loss of generality, we only consider static output feedback. Suppose that for the discrete-time LTI system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k), \end{cases} \tag{47}$$

where $y(k) \in \mathbb{R}^p$ is the measured output, we have designed a stabilizing static output $u = Ky$. Here, without causing confusion, we used the same notation K for the feedback gain. Then the Lyapunov equation is

$$(A + BKC)^T \bar{P}(A + BKC) - \bar{P} = -\bar{Q}. \tag{48}$$

Since we deal with the situation where only a quantized measured output $y(k)$ is available for the controller, the control input to the system is

$$u(k) = \mu_2 q_2 \left(\frac{K\mu_1 q_1(\frac{y(k)}{\mu_1})}{\mu_2} \right). \tag{49}$$

Then, for any fixed positive scalars μ_1 and μ_2, the closed-loop system composed of the system (47) and the controller (49) is given by

$$x(k+1) = (A + BKC)x(k) - B\bar{d}(k), \tag{50}$$

where

$$\bar{d}(k) = \mu_2 \left(\frac{KCx(k)}{\mu_2} - q_2 \left(\frac{K\mu_1 q_1(\frac{Cx(k)}{\mu_1})}{\mu_2} \right) \right). \tag{51}$$

Then it is easy to see that the discussion in Sections 4 and 5 remains valid if we replace $\|K\|$ with $\|K\|\|C\|$.

To summarize, we define

$$\begin{aligned} \bar{M} &= M_2 - \|K\|\|C\|\zeta(\Delta_1 + M_1), \\ \bar{\Delta} &= \|K\|\|C\|\zeta\Delta_1 + \Delta_2. \\ \bar{\alpha} &= \|(A + BKC)^T \bar{P} B\|, \quad \bar{\beta} = \|B^T \bar{P} B\|, \end{aligned} \tag{52}$$

and formulate the following theorem.

Theorem 2. *If $\bar{M}$ is large enough compared with $\bar{\Delta}$ in (52) so that*

$$\sqrt{\frac{\lambda_m(\bar{P})}{\lambda_M(\bar{P})}} \bar{M} > \frac{\bar{\alpha} + \sqrt{\bar{\alpha}^2 + \bar{\beta}\lambda_m(\bar{Q})}}{\lambda_m(\bar{Q})} \|K\|\|C\|\bar{\Delta} \tag{53}$$

holds with some positive scalar ζ, then there exists a hybrid quantized static output feedback strategy assuring that the system (50) is globally asymptotically stable.

■

Remark 4. Note that the condition (32) (resp. (53)) is given in terms of P (resp. $\bar{P}$), Q (resp. $\bar{Q}$), K and the quantizer parameters. Although in this paper we have focused our attention on adjusting the quantizer parameters, we may have to adjust P (resp. $\bar{P}$), Q (resp. $\bar{Q}$) for fixed K, or may have to change the feedback again K, so that the condition (32) (or (53)) is satisfied. Such kind of controller design constitutes an open problem for future research.

8. Conclusion

We have proposed a hybrid stabilization strategy for discrete-time LTI systems via state feedback (or output feedback) where both states (or measured outputs) and control input signals are quantized. In our problem formulation, the quantizers have a general form and their parameters are updated at discrete time instants. Therefore, the control method is practical in real computer-based applications. We also note that the extension to H_∞ feedback control systems was considered in the case of a single quantized signal (a quantized state or a quantized measured output) (Zhai *et al.*, 2005), and our future research includes the extension to H_∞ feedback control systems with more than two quantized signals.

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