OBSERVER DESIGN USING A PARTIAL NONLINEAR OBSERVER CANONICAL FORM

KLAUS RÖBENACK*, ALAN F. LYNCH**

* Technische Universität Dresden
Department of Mathematics, Institute of Scientific Computing
Mommsenstr. 13, D–01062 Dresden, Germany
e-mail: klaus@roebenack.de

** University of Alberta
Department of Electrical and Computer Engineering
Edmonton AB T6G 2V4, Canada
e-mail: alanl@ieee.org

This paper proposes two methods for nonlinear observer design which are based on a partial nonlinear observer canonical form (POCF). Observability and integrability existence conditions for the new POCF are weaker than the well-established nonlinear observer canonical form (OCF), which achieves exact error linearization. The proposed observers provide the global asymptotic stability of error dynamics assuming that a global Lipschitz and detectability-like condition holds. Examples illustrate the advantages of the approach relative to the existing nonlinear observer design methods. The advantages of the proposed method include a relatively simple design procedure which can be broadly applied.

Keywords: observer design, canonical form, detectability

1. Introduction

We consider the observer design problem for a SISO system

$$\dot{x} = f(x) + g(x,u), \quad y = h(x) \tag{1}$$

with smooth vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and smooth output functions $h : \mathbb{R}^n \to \mathbb{R}$. Exact error linearization is a well-established observer design method based on an observer canonical form (OCF) which yields linear time-invariant error dynamics in some state coordinates. Since the initial work in (Bestle and Zeitz, 1983; Krener and Isidori, 1983), many variations on and extensions to this design method have been proposed (Kazantzis and Kravaris, 1998; Krener and Respondek, 1985; Krener *et al.*, 1991; Krener and Xiao, 2002, Lynch and Bortoff, 2001; Marino and Tomei, 1995; Phelps, 1991, Respondek *et al.*, 2004; Rudolph and Zeitz, 1994; Wang and Lynch, 2005;2006; Xia and Gao, 1988;1989.) In the single-output case, the aforementioned work relies on the assumption

$$\dim \operatorname{span}\{\mathrm{d}h, \mathrm{d}L_f h, \dots, \mathrm{d}L_f^{n-1} h\}(x) = n \tag{2}$$

for all x in a suitable set. The function $L_f h = \frac{\partial h}{\partial x} f$ in (2) is the *Lie derivative* of h along f. Repeated Lie derivatives are defined as $L_f^k h = L_f(L_f^{k-1} h)$, $k \geq 1$ with $L_f^0 h = h$. The *differential* or *gradient* of a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\mathrm{d}\lambda$ and has a local coordinate description $\mathrm{d}\lambda = \frac{\partial \lambda}{\partial x} = (\frac{\partial \lambda}{\partial x_1}, \dots, \frac{\partial \lambda}{\partial x_n})$. The condition (2) ensures a form of observability for the unforced system (Hermann and Krener, 1977), and is necessary to ensure the existence of the OCF (Krener and Isidori, 1983). It is well known that OCF-based methods can be difficult to apply due to restrictive existence conditions. Also, the condition (2) does not always hold globally or even on a sufficiently large set to avoid a singular observer gain in many canonical form designs. In an effort to address these drawbacks, we propose an observer based on a *partial nonlinear observer canonical form* (POCF) which requires a weaker condition

$$\dim \operatorname{span}\{\mathrm{d}h, \mathrm{d}L_f h, \dots, \mathrm{d}L_f^{r-1} h\}(x) = r, \quad 1 \leq r < n \tag{3}$$

to hold for all x in a suitable set. Additionally, less restrictive integrability conditions than those for an OCF will be required. To ensure the convergence of the estimate error, we impose Lipschitz and detectability-like conditions.

Jo and Seo (2002) also consider observer design with the weaker observability condition (3). They propose an observer design based on

$$\dot{z}_0 = A_0 z_0 + \gamma_0(y,u), \tag{4a}$$

$$\dot{z}_{\bar{0}} = A_{\bar{0}0} z_0 + f_{\bar{0}}(y, z_{\bar{0}}) + \gamma_{\bar{0}}(y,u), \tag{4b}$$

$$y = c_0^T z_0, \tag{4c}$$

where $A_0 \in \mathbb{R}^{r\times r}$ and $c_0 \in \mathbb{R}^{r\times 1}$ are in a dual Brunovsky form (Brunovsky, 1970):

$$A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$c_0^T = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5}$$

The system (4) is divided into two parts: the first subsystem (4a) is isolated from the second one and is in an OCF. On the other hand, the second subsystem (4b) contains the term $f_{\bar{0}}$ which allows for a nonlinear dependence on both the second subsystem state $z_{\bar{0}}$ and the output. The output depends linearly on the first subsystem state z_0. Although the existence conditions for (4) are weaker than the OCF, in this paper we propose a POCF which exists under less restrictive conditions and is suitable for observer design. Two observer designs based on POCF coordinates are proposed. The first design has an advantage of a simpler gain expression. The second design leads to a simpler error convergence proof but involves a more complicated gain calculation.

This paper is organized as follows: Section 2 presents the existence conditions for the POCF. Section 3 presents two observers and a theorem for the global asymptotic convergence of their error dynamics. Section 4 presents examples.

2. Partial Nonlinear Observer Canonical Form (POCF)

First, we investigate the existence conditions for a diffeomorphism T transforming (1) into a partial nonlinear observer canonical form (POCF) of index $r \in \{1,\ldots,n-1\}$:

$$\dot{z} = A\,z + \alpha(y, z_{r+1},\ldots,z_n,u), \tag{6a}$$

$$y = c^T z, \tag{6b}$$

with $z = (z_1,\ldots,z_n)^T$, and $\alpha = \alpha_1 \frac{\partial}{\partial z_1} + \cdots + \alpha_n \frac{\partial}{\partial z_n}$ is a smooth vector field. The matrix $A \in \mathbb{R}^{n\times n}$ and the vector $c \in \mathbb{R}^{n\times 1}$ have the form

$$A = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{and} \quad c^T = \left(\begin{array}{c|c} c_0^T & 0 \end{array}\right),$$

where c_0 and A_0 are defined in (5).

We recall the following result on simultaneous rectification:

Theorem 1 (Nijmeijer and van der Schaft, 1990, Thm. 2.36)**.** *Let $X_1,\ldots,X_r$ be linearly independent vector fields defined on a neighbourhood of $\xi_0 \in \mathbb{R}^n$. Suppose that on a neighbourhood $U \subseteq \mathbb{R}^n$ of ξ_0*

$$[X_i, X_j] = 0, \quad 1 \le i, \quad j \le r.$$

Then there exist coordinates $(x_1,\ldots,x_n)$ defined on U such that on U

$$X_i = \frac{\partial}{\partial x_i}, \quad 1 \le i \le r.$$

We remark that when applying Theorem 1 later we will choose $n-r$ linearly independent vector fields X_i, $r+1 \le i \le n$ to X_i, $1 \le i \le r$ such that about ξ_0

$$[X_i, X_j] = 0, \quad 1 \le i, \quad j \le n.$$

This choice is nonunique and affects the expressions for the system in the new coordinates. The observer design method presented in (Jo and Seo, 2002) imposes additional constraints on the choice of X_i, $r+1 \le i \le n$, which are not required here. These additional constraints can limit the applicability of that approach.

In order to define the POCF, we need to define the so-called *starting vector field*. If $r < n$, the matrix

$$Q_r = \begin{pmatrix} \mathrm{d}h \\ \vdots \\ \mathrm{d}L_f^{r-1} h \end{pmatrix} \tag{7}$$

is called the *reduced observability matrix*. When $n = r$, we call (7) the *observability matrix*. A smooth solution v of

$$Q_r \cdot v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} =: e_r \in \mathbb{R}^r \tag{8}$$

is called the *starting vector field*. Before giving sufficient conditions for the existence of the POCF (6), we define some notation. The *Lie bracket* of two vector fields f and g is defined as $[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$. Repeated Lie brackets are defined as $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$, $k \ge 1$ with $\mathrm{ad}_f^0 g = 0$.

Theorem 2. *There exists a diffeomorphism $T : U \to \mathbb{R}^n$ defined on a neighbourhood U of x_0 transforming* (1) *into POCF* (6) *of index r if*

(C1) $\operatorname{rank} Q_r = r$,

(C2) $[\mathrm{ad}_f^i v, \mathrm{ad}_f^j v] = 0, \quad 0 \le i, \quad j \le r-1,$

(C3) $[g, \mathrm{ad}_\mathrm{f}^\mathrm{i} \mathrm{v}] = 0, \quad 0 \le \mathrm{i} \le \mathrm{r}-2,$

in some neighbourhood of x_0. The diffeomorphism T is global if the conditions C1–C3 hold on $\mathbb{R}^n$ and, in addition,

(C4) $ad_{-f}^i v$, $0 \le i, j \le r-1$ *are complete vector fields.*

Proof. The proof is divided into two parts. In Part A we show that there exists a change of coordinates $\zeta = \Psi(x)$ which transforms (1) into

$$\dot{\zeta} = A\zeta + \eta(\zeta_r, \zeta_{r+1}, \dots, \zeta_n, u), \tag{9a}$$

$$y = c^T \zeta + \beta(\zeta_{r+1}, \dots, \zeta_n), \tag{9b}$$

with a smooth vector field $\eta = \eta_1 \frac{\partial}{\partial \zeta_1} + \cdots + \eta_n \frac{\partial}{\partial \zeta_n}$, a smooth map β, and $\zeta = (\zeta_1, \dots, \zeta_n)^T$. In Part B we construct a second coordinate system in which $\beta \equiv 0$.

Part A: Assume that the conditions C1–C3 of Theorem 2 are satisfied. The condition C1 implies that (8) has a solution v defined on some neighbourhood of $x_0 \in \mathbb{R}^n$. Equation (8) can be rewritten as

$$L_v L_f^i h = \begin{cases} 0 & \text{for} \quad 0 \le i \le r-2, \\ 1 & \text{for} \quad i = r-1. \end{cases}$$

From (Isidori, 1995, Lem. 4.1.2), this implies that

$$\begin{pmatrix} \mathrm{d}h \\ \vdots \\ \mathrm{d}L_f^{r-1} h \end{pmatrix} \begin{pmatrix} v & \mathrm{ad}_{-f} v & \cdots & \mathrm{ad}_{-f}^{r-1} v \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & * \\ 0 & \iddots & \iddots & \vdots \\ 1 & * & \cdots & * \end{pmatrix} \tag{10}$$

in a neighbourhood of x_0. Therefore, the vector fields $v, \mathrm{ad}_f v, \dots, \mathrm{ad}_f^{r-1} v$ are linearly independent in some neighbourhood of x_0. Using the condition C2 and Theorem 1, we deduce that there exists a local diffeomorphism $\zeta = \Psi(x)$ such that

$$\Psi_* \mathrm{ad}_{-f}^i v = \frac{\partial}{\partial \zeta_{i+1}}, \quad 0 \le i \le r-1, \tag{11}$$

where $\Psi_* = \partial \Psi / \partial x$. For clarity, the representations of f, g, and h in the ζ-coordinates are denoted by

$$\bar{f}(\zeta) = \Psi_* f(x) \mid_{x=\Psi^{-1}(\zeta)},$$
$$\bar{g}(\zeta, u) = \Psi_* g(x, u) \mid_{x=\Psi^{-1}(\zeta)},$$
$$\bar{h}(\zeta) = h(x) \mid_{x=\Psi^{-1}(\zeta)} .$$

Owing to (10), we have

$$L_{ad_{-f}^i v} h = \frac{\partial \bar{h}}{\partial \zeta_{i+1}} = \begin{cases} 0 & \text{for} \quad 0 \le i \le r-2, \\ 1 & \text{for} \quad i = r-1. \end{cases}$$

Therefore, the gradient of $\bar{h}$ has the form

$$\frac{\partial \bar{h}}{\partial \zeta} = \begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}, \tag{12}$$

where the leading one on the right-hand-side of (12) appears in the r-th column. Hence, in the ζ-coordinates the output map $\bar{h}$ has the form given in (9b). Next, we consider the drift vector field

$$\bar{f}(\zeta) = \bar{f}_1(\zeta) \frac{\partial}{\partial \zeta_1} + \cdots + \bar{f}_n(\zeta) \frac{\partial}{\partial \zeta_n}.$$

Due to (11), for $1 \le i \le r-1$ we have

$$\begin{aligned} \frac{\partial}{\partial \zeta_{i+1}} &= \Psi_* \mathrm{ad}_{-f}^i v \\ &= \Psi_* [-f, \mathrm{ad}_{-f}^{i-1} v] \\ &= [-\Psi_* f, \Psi_* \mathrm{ad}_{-f}^{i-1} v] \\ &= [-\Psi_* f, \frac{\partial}{\partial \zeta_i}] \\ &= [-\bar{f}, \frac{\partial}{\partial \zeta_i}] \\ &= \sum_{j=1}^n \frac{\partial \bar{f}_j}{\partial \zeta_i} \frac{\partial}{\partial \zeta_j}. \end{aligned} \tag{13}$$

Comparing both sides of (13) yields

$$\begin{aligned} \frac{\partial \bar{f}_j}{\zeta_i} &= 0 \ \text{for} \quad 1 \le j \le n, \ j = i+1, \\ & \qquad\qquad 1 \le i \le r-1, \\ \frac{\partial \bar{f}_{i+1}}{\zeta_i} &= 1 \ \text{for} \quad 1 \le i \le r-1. \end{aligned} \tag{14}$$

This means that the Jacobian matrix of $\bar{f}$ has the form

$$\frac{\partial \bar{f}}{\partial \zeta}(\zeta) = \left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & * & * & \cdots & * \\ 1 & \ddots & \vdots & \vdots & \vdots & & \vdots \\ & \ddots & 0 & \vdots & \vdots & & \vdots \\ 0 & & 1 & * & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & * & \cdots & * \end{array}\right). \tag{15}$$

Finally, we consider the input-dependent vector field $\bar{g}$. Because of the condition C3 and (11), for $0 \le i \le r-2$ we have

$$\begin{aligned} 0 &= \Psi_*[g, \mathrm{ad}^i_{-f} v] \\ &= [\Psi_* g, \Psi_* \mathrm{ad}^i_{-f} v] \\ &= \left[\bar{g}, \frac{\partial}{\partial \zeta_{i+1}}\right] \\ &= -\sum_{j=1}^{n} \frac{\partial \bar{g}_j}{\zeta_{i+1}} \frac{\partial}{\partial \zeta_j}. \end{aligned}$$

This implies

$$\frac{\partial \bar{g}_j}{\partial \zeta_{i+1}} = 0, \quad 1 \le j \le n, \quad 0 \le i \le r-2. \tag{16}$$

Hence, the Jacobian matrix of $\bar{g}$ looks like

$$\frac{\partial \bar{g}}{\partial \zeta}(\zeta, u) = \left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & * & \cdots & * \end{array}\right). \tag{17}$$

From (14) and (16) (or, equivalently, (15) and (17)), we can conclude that the right-hand side of the transformed system has the form (9).

Part B: In this part we construct a second change of coordinates transforming (9) into (6). Let $z = \Phi(\zeta)$ be a global diffeomorphism defined by

$$\begin{aligned} z_i &= \zeta_i, \quad i \ne r, \quad 1 \le i \le n, \\ z_r &= \zeta_r + \beta(\zeta_{r+1}, \dots, \zeta_n). \end{aligned}$$

From (9b), we have (6b):

$$y = c^T z.$$

The dynamics transform into (6a) with

$$\begin{aligned} &\alpha_i(y, z_{r+1}, \dots, z_n, u) \\ &\quad = \eta_i\big(z_r - \beta(z_{r+1}, \dots, z_n), z_{r+1}, \dots, z_n, u\big), \\ &\quad\quad i \ne r, \quad 1 \le i \le n, \\ &\alpha_r(y, z_{r+1}, \dots, z_n, u) \\ &\quad = \eta_r\big(z_r - \beta(z_{r+1}, \dots, z_n), z_{r+1}, \dots, z_n, u\big) \\ &\quad\quad + \sum_{j=r+1}^{n} \frac{\partial \beta}{\partial \zeta_j} \eta_j(\zeta, u)\Big|_{\zeta = \Phi^{-1}(z)}. \end{aligned}$$

Therefore, the diffeomorphism T which transforms (1) into the POCF (6) is a composition of the transformations given in Part A and B: $T = \Phi \circ \Psi$. Part A fixes the dependence of the system on the first r coordinates without specifying the dependence on the remaining $n - r$ coordinates. Part B only changes the dependence in the r-th coordinates to ensure that the output equals z_r.

If the conditions C1–C3 hold globally, the condition C4 on the completeness of the vector fields implies the existence of a global diffeomorphism (Respondek, 1986). ■

We remark that, if $r = n$, the conditions in Theorem 2 are the same as those of the OCF (Krener and Isidori, 1983). Evidently, for $r < n$ the proposed existence conditions are satisfied by a larger class of systems than those admitting an OCF.

When $n = 2$, we can only have a POCF of index $r = 1$. In this case, only the condition C1 (i.e., $\mathrm{d}h \ne 0$) must be checked since C2 and C3 are always satisfied.

As is mentioned in the proof of Theorem 2, the condition C1 implies that a solution of (8) exists but is not unique. This nonuniqueness can be used to simplify the vector fields $\mathrm{ad}^i_{-f} v, 1 \le i \le r-1$. Simpler expressions for these vector fields lead to a less complex observer design. A particular solution of (8) is given by $v = Q_r^+ e_r$, where $Q_r^+ = (Q_r^T Q_r)^{-1} Q_r^T$ denotes the Moore-Penrose inverse (Moore, 1920).

3. Observer Design and Error Convergence

We consider two observer designs which are based on the POCF (6). The first design has an advantage of a simpler expression for its gain. The second design requires the knowledge of the POCF coordinates to compute its gain.

When discussing observers and their convergence, it is convenient to introduce an alternative notation for the POCF. We split (6) into two subsystems:

$$\begin{aligned} \dot{z}_1 &= A_0 z_1 + \alpha_1(y, z_2, u), \\ \dot{z}_2 &= \alpha_2(y, z_2, u), \\ y &= c_0^T z_1, \end{aligned}$$

where z_1 denotes the first r components of z, and z_2 stands for the last $n - r$ components of z. Similarly, α_1 denotes the first r components of α, and α_2 signifies the last $n - r$ components of α.

3.1. Observer Design No. 1.

We consider a Luenberger-like observer structure

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x}, u) + k(\hat{x})\big(y - h(\hat{x})\big), \tag{18}$$

where the gain vector k depends on the estimated state alone. Assuming that the system (1) satisfies the conditions of Theorem 2, we can express the observer (18) in the POCF coordinates

$$\begin{pmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \end{pmatrix} = \begin{pmatrix} A_0\hat{z}_1 + \alpha_1(\hat{y}, \hat{z}_2, u) \\ \alpha_2(\hat{y}, \hat{z}_2, u) \end{pmatrix} + (S'(\hat{z}))^{-1} k\big(S(\hat{z})\big)\big(y - h(\hat{x})\big), \tag{19}$$

where $S = T^{-1}$, $S' = \partial x/\partial z$ and $\hat{y} = c_0^T\hat{z}_1$. We consider the choice

$$k\big(S(\hat{z})\big) = S'(\hat{z}) \begin{pmatrix} l \\ 0 \end{pmatrix} \tag{20}$$

with $l = (p_0, \dots, p_{r-1})^T$, and below, in Section 3.3, we will appropriately assign the roots of

$$\det\big(\lambda I - (A_0 - lc_0^T)\big) = p_0 + p_1\lambda + \dots + p_{r-1}\lambda^{r-1} + \lambda^r. \tag{21}$$

Substituting (20) into (19), we obtain

$$\dot{\hat{z}}_1 = A_0\hat{z}_1 + \alpha_1(\hat{y}, \hat{z}_2, u) + l(y - c_0^T\hat{z}_1), \tag{22a}$$

$$\dot{\hat{z}}_2 = \alpha_2(\hat{y}, \hat{z}_2, u). \tag{22b}$$

The estimation error $\tilde{z} = z - \hat{z}$ of this observer is governed by

$$\dot{\tilde{z}}_1 = (A_0 - lc_0^T)\tilde{z}_1 + \alpha_1(y, z_2, u) - \alpha_1(\hat{y}, \hat{z}_2, u), \tag{23a}$$

$$\dot{\tilde{z}}_2 = \alpha_2(y, z_2, u) - \alpha_2(\hat{y}, \hat{z}_2, u). \tag{23b}$$

An observer is typically implemented in the original x-coordinates and, ideally, to simplify the design procedure, the gain k can be computed without requiring expressions for the POCF coordinates or related functions α_1 and α_2. Since S is the inverse of T, we can rewrite (11) in the form

$$\mathrm{ad}^i_{-f} v(x) = S'\big(T(x)\big)\, e_{i+1}, \quad 0 \le i \le r-1.$$

Hence from (20) we have a simple expression for the observer gain:

$$k(\hat{x}) = p_0 v(\hat{x}) + p_1 \mathrm{ad}_{-f} v(\hat{x}) + \dots + p_{r-1}\mathrm{ad}^{r-1}_{-f} v(\hat{x}). \tag{24}$$

3.2. Observer Design No. 2.

If we choose the observer structure

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x}, u) + k(\hat{x}, y, u) \tag{25}$$

and require a cascade or triangular form error dynamics

$$\dot{\tilde{z}}_1 = (A_0 - lc_0^T)\tilde{z}_1 + \alpha_1(y, z_2, u) - \alpha_1(y, \hat{z}_2, u), \tag{26a}$$

$$\dot{\tilde{z}}_2 = \alpha_2(y, z_2, u) - \alpha_2(y, \hat{z}_2, u), \tag{26b}$$

then this implies that in the z-coordinates the observer is

$$\dot{\hat{z}}_1 = A_0\hat{z}_1 + \alpha_1(y, \hat{z}_2, u) + l(y - c_0^T\hat{z}_1), \tag{27a}$$

$$\dot{\hat{z}}_2 = \alpha_2(y, \hat{z}_2, u), \tag{27b}$$

and the gain in (25) is

$$k\big(S(\hat{z}), y, u\big) = S'(\hat{z})\big(\alpha(y, \hat{z}_2, u) - \alpha(\hat{y}, \hat{z}_2, u) + lc_0^T\tilde{z}_1\big), \tag{28}$$

where the constant gain vector l is chosen below, in Section 3.3, to assign the roots of (21).

Comparing (22) and (27), we remark that the observers differ in that the second one uses y in place of $\hat{y}$. From this one might expect that the second design uses more exact system information and might lead to better convergence.

3.3. Error Dynamics Convergence.

Next, we demonstrate the convergence of the observers (18), (24) and (25), (28). We treat the convergence of the observers in separate theorems and consider (18), (24) first.

3.3.1. Observer Design No. 1.

We begin with the following assumptions:

(A1) The input u is bounded, i.e., there exists a positive constant γ_0 such that $|u(t)| \le \gamma_0, t \ge 0$.

(A2) The map α_1 is globally Lipschitz in y and z_2, uniformly in u, i.e., there exist positive constants γ_1, γ_2 such that

$$\|\alpha_1(y, z_2, u) - \alpha_1(\hat{y}, \hat{z}_2, u)\| \le \gamma_1\|\tilde{y}\| + \gamma_2\|\tilde{z}_2\|$$

for all $y, \hat{y} \in \mathbb{R}$, $z_2, \hat{z}_2 \in \mathbb{R}^{n-r}$, and any bounded u.

As in (Amicucci and Monaco, 1998), we require a steady-state solution property of the system. The next assumption is the uniform robust steady-state solution property with respect to y:

(A3) There exist a positive definite matrix $P_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ and positive constants γ_3, γ_4 such that for $V_2(\tilde{z}_2) = \tilde{z}_2^T P_2 \tilde{z}_2$ we have

$$\begin{aligned} \frac{\partial V_2(\tilde{z}_2)}{\partial \tilde{z}_2}&\big(\alpha_2(y, z_2, u) - \alpha_2(\hat{y}, \hat{z}_2, u)\big) \\ &= 2\tilde{z}_2^T P_2\big(\alpha_2(y, z_2, u) - \alpha_2(\hat{y}, \hat{z}_2, u)\big) \\ &\le \gamma_3\|\tilde{y}\|^2 - \gamma_4\|\tilde{z}_2\|^2 \end{aligned} \tag{29}$$

for all $y, \hat{y} \in \mathbb{R}$, $z_2, \hat{z}_2 \in \mathbb{R}^{n-r}$, and any bounded u.

The function V_2 is also called an exponential-decay output-to-state stable (OSS) Lyapunov function (Sontag and Wang, 1997).

Before stating the convergence theorem, we introduce a lemma from (Röbenack and Lynch, 2004) which is a slightly different form of a result in (Gauthier *et al.*, 1992).

Lemma 1. *Given* A_0 *and* c_0 *defined in* (5), *consider the Lyapunov equation*

$$A_0^T P(\theta) + P(\theta) A_0 + \theta P(\theta) = c_0 c_0^T, \tag{30}$$

where θ *is a positive number and* $P \in \mathbb{R}^{r\times r}$. *Then there exists* $\bar\theta > 0$ *such that the Lyapunov equation* (30) *has a positive definite solution*

$$P(\theta) > 0 \quad \text{with} \quad P^2(\theta) \le P(\theta), \quad \forall \theta \ge \bar\theta\ . \tag{31}$$

Proof. It can directly be verified that the (i,j)-th entry of P satisfying (30) is given by

$$p_{ij} = \frac{(-1)^{i+j}}{\theta^{2r-i-j+1}} \cdot \frac{(2r-i-j)!}{(r-i)!\,(r-j)!}, \quad 1 \le i,j \le r. \tag{32}$$

Moreover, this solution of (30) is unique and positive definite. Therefore, all eigenvalues of P are real and positive. Due to (32), all entries of P converge to 0 as $\theta \to \infty$. Hence, the eigenvalues of P also converge to 0 as $\theta \to \infty$ and there exists $\bar\theta > 0$ such that the eigenvalues of P are less than 1 for all $\theta \ge \bar\theta$. ∎

Theorem 3. *Consider the system* (1) *together with the observer* (18) *and the observer gain* (24). *Assume that the conditions C1–C4 hold and, under Assumptions A1–A3, there exists a vector* $l \in \mathbb{R}^r$ *such that*

$$\lim_{t\to\infty} \|\hat x(t) - x(t)\| = 0$$

for all initial values $x(0)$ *and* $\hat x(0)$ *of* (1) *and* (18), *respectively.*

Proof. Our proof is based on the work (Gauthier *et al.*, 1992). Assuming that the conditions C1–C4 hold, convergence can be analysed in the POCF coordinates. We have to show that the equilibrium $\tilde z = 0$ of (23) is globally asymptotically stable. Let $P \in \mathbb{R}^{r\times r}$ be a positive definite matrix which will be specified later, and take the positive definite matrix P_2 from Assumption A3. Then the candidate Lyapunov function

$$V(\tilde z_1, \tilde z_2) = V_1(\tilde z_1) + V_2(\tilde z_2)$$

with

$$V_1(\tilde z_1) = \tilde z_1^T P \tilde z_1 \quad \text{and} \quad V_2(\tilde z_2) = \tilde z_2^T P_2 \tilde z_2$$

is positive definite and radially unbounded. The time derivative of V_1 along (23a) is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_1(\tilde z_1)\Big|_{(23a)} &= \tilde z_1^T \left[(A_0 - l c_0^T)^T P + P(A_0 - l c_0^T)\right] \tilde z_1 \\ &\quad + 2\tilde z_1^T P\big[\alpha_1(y, z_2, u) - \alpha_1(\hat y, \hat z_2, u)\big]. \end{aligned}\tag{33}$$

We choose the gain vector as

$$l = \frac{\nu}{2} P^{-1} c_0 \quad \text{with} \quad \nu > 0. \tag{34}$$

Hence we have

$$(A_0 - l c_0^T)^T P + P(A_0 - l c_0^T) = A_0^T P + P A_0 - \nu\, c_0 c_0^T. \tag{35}$$

Using A2, we obtain

$$\begin{aligned} & 2\tilde z_1^T P\big[\alpha_1(y, z_2, u) - \alpha_1(\hat y, \hat z_2, u)\big] \\ &\le 2\left|\tilde z_1^T P\big[\alpha_1(y, z_2, u) - \alpha_1(\hat y, \hat z_2, u)\big]\right| \\ &\le 2\,\|P\tilde z_1\| \cdot \|\alpha_1(y, z_2, u) - \alpha_1(\hat y, \hat z_2, u)\| \\ &\le 2\,\|P\tilde z_1\| \cdot (\gamma_1 \|\tilde y\| + \gamma_2 \|\tilde z_2\|) \\ &\le 2\gamma_1\,\|P\tilde z_1\| \cdot \|c_0^T \tilde z_1\| + 2\gamma_2\,\|P\tilde z_1\| \cdot \|\tilde z_2\| \qquad (36) \\ &\le \gamma_1^2 \tilde z_1^T P^2 \tilde z_1 + \tilde z_1^T c_0 c_0^T \tilde z_1 + \frac{\gamma_2^2}{\mu} \tilde z_1^T P^2 \tilde z_1 + \mu\, \tilde z_2^T \tilde z_2 \qquad (37) \\ &\le \left(\gamma_1^2 + \frac{\gamma_2^2}{\mu}\right) \tilde z_1^T P^2 \tilde z_1 + \tilde z_1^T c_0 c_0^T \tilde z_1 + \mu\, \tilde z_2^T \tilde z_2 \qquad (38) \end{aligned}$$

for all $\mu > 0$. Going from (36) to (37) we have used

$$ab \le (\delta a)^2 + (b/\delta)^2, \quad \forall \delta \in \mathbb{R}\backslash\{0\}, a, b \in \mathbb{R}\ .$$

Combining (33), (35), and (38) results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_1(\tilde z_1)\Big|_{(23a)} &\le \tilde z_1^T \left(A_0^T P + P A_0\right) \tilde z_1 + \mu \tilde z_2^T \tilde z_2 \\ &\quad + \tilde z_1^T \left(\left(\gamma_1^2 + \frac{\gamma_2^2}{\mu}\right) P^2 - (\nu - 1) c_0 c_0^T\right) \tilde z_1. \end{aligned}\tag{39}$$

Using Assumption A3, a bound on the time derivative of V_2 along (23b) is given by (29):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_2(\tilde z_2)\Big|_{(23b)} &\le \gamma_3 \|\tilde y\|^2 - \gamma_4 \|\tilde z_2\|^2 \\ &\le \gamma_3 \tilde z_1^T c_0 c_0^T \tilde z_1 - \gamma_4 \tilde z_2^T \tilde z_2. \end{aligned}\tag{40}$$

From (39) and (40) we collect the terms with $\|\tilde z_2\|^2$:

$$(\mu - \gamma_4)\|\tilde z_2\|^2. \tag{41}$$

This quadratic form is negative definite for any $\mu \in (0, \gamma_4)$. Next, we collect the terms with $\tilde{z}_1$ occurring in (39) and (40) and obtain

$$\tilde{z}_1^T \Big[A_0^T P + P A_0 - (\nu - 1 - \gamma_3) c_0 c_0^T + \big(\gamma_1 + \frac{\gamma_2^2}{\mu}\big) P^2 \Big] \tilde{z}_1. \quad (42)$$

Take $\bar{\theta}$ from Lemma 1 and choose

$$\theta > \max \left\{ \bar{\theta}, \gamma_1 + \frac{\gamma_2^2}{\mu} \right\} \quad \text{and} \quad \nu > \gamma_3.$$

Using Lemma 1, the matrix P is the unique solution of

$$A_0^T P(\theta) + P(\theta) A_0 + \theta P(\theta) = c_0 c_0^T.$$

Then the quadratic form (42) can be bounded as

$$\begin{aligned} &\tilde{z}_1^T \Big[A_0^T P + P A_0 - (\nu - 1 - \gamma_3) c_0 c_0^T + \Big(\gamma_1 + \frac{\gamma_2^2}{\mu}\Big) P^2 \Big] \tilde{z}_1 \\ &\le \tilde{z}_1^T \left[\left(\gamma_1 + \frac{\gamma_2^2}{\mu}\right) P^2 - \theta P - (\nu - \gamma_3) c_0 c_0^T \right] \tilde{z}_1 \\ &\le \tilde{z}_1^T \left[\left(\gamma_1 + \frac{\gamma_2^2}{\mu}\right) P^2 - \theta P \right] \tilde{z}_1 \\ &\le \tilde{z}_1^T \left[\left(\gamma_1 + \frac{\gamma_2^2}{\mu} - \theta P\right) P \right] \tilde{z}_1, \end{aligned} \quad (43)$$

where we employed (31). Since (41) and (43) are both negative definite, we conclude that

$$\dot{V}(\tilde{z}_1, \tilde{z}_2) \Big|_{(23)} < 0 \quad \text{for} \quad (\tilde{z}_1, \tilde{z}_2) \neq (0, 0).$$

Therefore, V is a Lyapunov function of (23) and the equilibrium $(\tilde{z}_1, \tilde{z}_2) = (0, 0)$ is globally asymptotically stable. ∎

3.3.2. Observer Design No. 2. We require Assumption A1 and the following two modified versions of Assumptions A2 and A3:

(A4) The map α_1 is globally Lipschitz in z_2 uniformly in y and u, i.e., there exists a positive constant $\gamma_2 > 0$ such that

$$\|\alpha_1(y, z_2, u) - \alpha_1(y, \hat{z}_2, u)\| \le \gamma_2 \|\tilde{z}_2\|$$

for all $y \in \mathbb{R}$, $z_2, \hat{z}_2 \in \mathbb{R}^{n-r}$, and any bounded u.

(A5) There exist a positive definite matrix $P_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ and a positive constant γ_4 such that for $V_2(\tilde{z}_2) = \tilde{z}_2^T P_2 \tilde{z}_2$ we have

$$\begin{aligned} &\frac{\partial V_2(\tilde{z}_2)}{\partial \tilde{z}_2} \left(\alpha_2(y, z_2, u) - \alpha_2(y, \hat{z}_2, u) \right) \\ &= 2 \tilde{z}_2^T P_2 \left(\alpha_2(y, z_2, u) - \alpha_2(y, \hat{z}_2, u) \right) \le -\gamma_4 \|\tilde{z}_2\|^2 \end{aligned} \quad (44)$$

for all $y \in \mathbb{R}$, $z_2, \hat{z}_2 \in \mathbb{R}^{n-r}$, and any bounded u.

The convergence result for the error dynamics (26) is given by the following theorem, whose proof is based on Theorem 3.

Theorem 4. *Consider the system* (1) *together with the observer* (25)*, where the observer gain is given by* (28)*. Assume that the conditions C1–C4 hold. Under Assumptions A1, A4, and A5, there exists a vector $l \in \mathbb{R}^r$ such that*

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0$$

for all initial values $x(0)$ and $\hat{x}(0)$ of (1) *and* (25)*, respectively.*

Proof. The proof is identical to that of Theorem 3 with $\gamma_1 = \gamma_3 = 0$. Hence we require

$$\theta > \max \left\{ \bar{\theta}, \frac{\gamma_2^2}{\mu} \right\}, \quad \text{and} \quad \nu > 2$$

and, as before, $\mu \in (0, \gamma_4)$. With the values of θ and ν satisfying these inequalities, we can compute l using (30) and (34). ∎

It is important to note that although the stability results in Theorem 3 and 4 are stated globally, following the results in (Gauthier *et al.*, 1992) or (Shim *et al.*, 2001), we can obtain semi-global stability results with weaker conditions, sufficient for most practical applications. In particular, we do not require a global Lipschitz assumption for a semi-global result.

4. Examples

4.1. Synchronous Machine. Neglecting damper windings, armature resistance, time derivatives of stator flux linkages and back-emf in stator voltage expressions, a synchronous motor can be expressed in state space form as follows (Birk and Zeitz, 1988; Keller, 1986; Mukhopadhyay and Malik, 1972):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= B_1 - A_1 x_2 - A_2 x_3 \sin x_1 - \frac{1}{2} B_2 \sin(2x_1), \\ \dot{x}_3 &= u - D_1 x_3 + D_2 \cos x_1, \\ y &= x_1, \end{aligned} \quad (45)$$

$$k(\hat{x},y)=\begin{pmatrix}(p_1-A_1)(y-\hat{x}_1)\\(p_0-A_1p_1+A_1^2)(y-\hat{x}_1)-A_2\hat{x}_3(\sin y-\sin\hat{x}_1)-B_2\ \left(\sin(2y)-\sin(2\hat{x}_1)\right)\\D_2(\cos y-\cos\hat{x}_1)\end{pmatrix}. \tag{48}$$

The measured output and the first state component x_1 denote the rotor position, x_2 is the rotor velocity, and x_3 is the field winding flux linkage. The control u is proportional to the voltage applied to field winding.

The observability matrix

$$Q_3(x) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -A_2x_3\cos x_1-B_2\cos(2x_1) & -A_1 & -A_2\sin x_1\end{pmatrix}$$

is not regular for $x_1\in\pi\mathbb{Z}$. The unique starting vector field for Q_3 satisfying (8) is

$$v(x)=\begin{pmatrix}0\\0\\-\dfrac{1}{A_2\sin x_1}\end{pmatrix},$$

which is not defined for $x_1\in\pi\mathbb{Z}$. Since $[\mathrm{ad}^1_{-f}v, \mathrm{ad}^2_{-f}v]\neq 0$, the integrability condition for the OCF is not fulfilled (Krener and Isidori, 1983). Further, adding an output transformation does not lead to an OCF.

We consider the observer design proposed in Section 3.1 with the index $r=2$. We remark that, in general, the proposed method allows for a range of choice for r. The reduced observability matrix Q_2 has the form

$$Q_2=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}.$$

A starting vector field satisfying (8) is $v=Q_2^+e_2=(0,1,0)^T$. This v results in $\mathrm{ad}_{-f}v=(1,-A_1,0)^T$. We supplement this vector with the vector $w_1=(0,0,1)^T$ so that the Jacobian matrix

$$S'(z)=\begin{pmatrix}0&1&0\\1&-A_1&0\\0&0&1\end{pmatrix}$$

is nonsingular.

We compute the transformations $x=S(z)$ and $z=T(x)$ that are linear:

$$\begin{aligned}x_1&=z_2\\x_2&=z_1-A_1z_2\\x_3&=z_3\end{aligned}\quad\text{and}\quad\begin{aligned}z_1&=A_1x_1+x_2,\\z_2&=x_1,\\z_3&=x_3.\end{aligned}$$

Applying this transformation to (45) yields

$$\begin{aligned}\begin{pmatrix}\dot{z}_1\\\dot{z}_2\\\dot{z}_3\end{pmatrix}&=\left(\begin{array}{cc|c}0&0&0\\1&0&0\\\hline 0&0&0\end{array}\right)\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\\&\quad+\underbrace{\begin{pmatrix}B_1-A_2z_3\sin z_2-\dfrac{B_2}{2}\sin(2z_2)\\-A_1z_2\\u-D_1z_3+D_2\cos z_2\end{pmatrix}}_{\alpha(z_2,z_3,u)},\\y&=z_2.\end{aligned}\tag{46}$$

The second subsystem has the form

$$\dot{z}_3=u-D_1z_3+D_2\cos z_2. \tag{47}$$

This system is linear if we consider the signals u and z_2 as time-dependent inputs. Its “unforced dynamics” have an asymptotically stable equilibrium at $z_3=0$ for $D_1=0.3222>0$. The observer gain (28) has the form (48). For the simulation parameters $A_1=0.2703$, $A_2=12.01$, $B_1=39.19$, $B_2=-48.04$, $D_1=0.3222$, $D_2=1.9$, and $u\equiv 1.933$ were used. The initial conditions are $x(0)=(0.8,0.1,10)^T$ and $\hat{x}(0)=(0,0,0)^T$ (all variables are per unit). The observer eigenvalues were placed at -10, i.e., $p_0=100$ and $p_1=20$. The simulation results are shown in Fig. 1. The slow convergence of the proposed observer is due to $\exp(-D_1t)$ resulting from the second subsystem (47).

It is important to note that the example does not admit an OCF (Krener and Isidori, 1983) or a partial nonlinear observer form (Jo and Seo, 2002). Also, extended Luenberger observer design leads to very large expressions (Birk and Zeitz, 1988). We remark that the observability condition (2) is only satisfied locally and there are advantages to not having the observer depend on the inverse of the observability matrix as this avoids singularities in the observer gain. This inverse appears in most high-gain designs and other related methods based on canonical forms. Finally, the example illustrates the computationally simple nature of the design.

4.2. Magnetic Levitation System. Under standard modelling assumptions, a one degree-of-freedom mag-

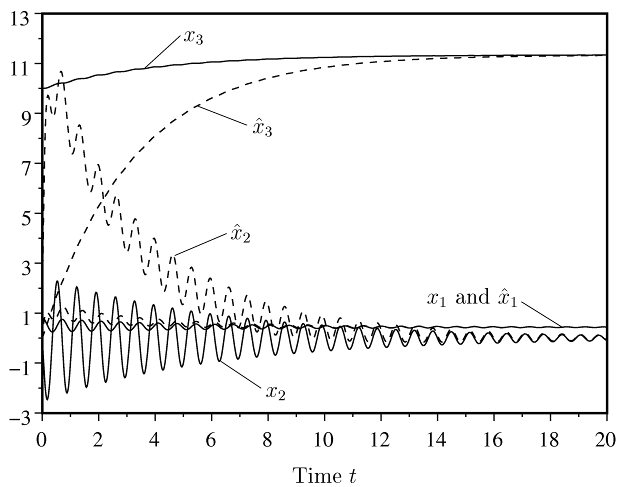

Fig. 1. Trajectories of the motor example.

netic levitation system can be modelled by

$$f(x)=\begin{pmatrix} \dfrac{x_1x_3}{x_2}-\dfrac{Rx_1x_2}{2\beta} \\ x_3 \\ g-\dfrac{\beta x_1^2}{mx_2^2} \end{pmatrix}, \quad g(x,u)=\begin{pmatrix} \dfrac{x_2}{2\beta} \\ 0 \\ 0 \end{pmatrix}u, \tag{49}$$
$$y=h(x)=x_2.$$

Here x_1 is the coil current, x_2 the shifted rotor position, x_3 the rotor velocity (Schweitzer *et al.*, 1994), and g, m, R, β are positive constants. As the rotor makes physical contact with the coil at $x_2 = c > 0$, we must have $x_2 \geq c$. An OCF does not exist for the system (49). This can be seen by first transforming the system to observable form coordinates $\xi_1 = \psi(x_2), \xi_2 = L_f\xi_1 = \psi'(x_2)x_3, \xi_3 = L_f^2\xi_2$, which include an output transformation denoted by ψ (Krener and Respondek, 1985). We transform the input vector field g into the observable form coordinates

$$\tilde{g}(x)=\frac{\partial\xi}{\partial x}g(x)=\begin{pmatrix} 0 \\ 0 \\ -\dfrac{x_1\psi'(x_2)}{mx_2} \end{pmatrix},$$

where $\tilde{g}$ is the representation of g in the $\xi = (\xi_1, \xi_2, \xi_3)^T$ coordinates. Since the Jacobian matrix $\frac{\partial\xi}{\partial x}$ has the form

$$\frac{\partial\xi}{\partial x}=\begin{pmatrix} 0 & \psi'(x_2) & 0 \\ 0 & \psi''(x_2)x_3 & \psi'(x_2) \\ * & * & * \end{pmatrix},$$

we necessarily have $\frac{\partial\xi_3}{\partial x_1} \neq 0$ for $\frac{\partial\xi}{\partial x}$ to be nonsingular. Since the starting vector in observable form coordinates is $v = \frac{\partial}{\partial\xi_3}$, we have

$$[v,\tilde{g}](x)=\begin{pmatrix} 0 \\ 0 \\ -\dfrac{\partial}{\partial\xi_3}\left(\dfrac{x_1\psi'(x_2)}{mx_2}\right) \end{pmatrix}\neq 0.$$

Therefore an OCF including an output transformation does not exist (Krener and Respondek, 1985).

We consider a transformation to the POCF of index $r = 2$. We have $v = Q_2^+e_2 = (0,0,1)^T$ and $\mathrm{ad}_{-f}v = (x_1/x_2, 1, 0)^T$. Defining the complete vector field $w_1 = (x_2, 0, 0)^T$ as the last column of the Jacobian matrix

$$S'(z)=\begin{pmatrix} 0 & x_1/x_2 & x_2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we ensure S' to be nonsingular for $x_2 \geq c$ and $[v, w_1] = 0$, $[\mathrm{ad}_{-f}v, w_1] = 0$. Letting $\Psi_v^t(x_0)$ denote the flow of the vector field v, we have

$$\Psi_v^{z_1}(x_0)=\begin{pmatrix} x_{10} \\ x_{20} \\ z_1 \end{pmatrix},$$

$$\Psi_{ad_{-f}v}^{z_2}(x_0)=\begin{pmatrix} \dfrac{x_{10}}{x_{20}}(z_2+x_{20}) \\ z_2+x_{20} \\ x_{30} \end{pmatrix},$$

$$\Psi_{w_1}^{z_3}(x_0)=\begin{pmatrix} x_{20}z_3+x_{10} \\ x_{20} \\ x_{30} \end{pmatrix}.$$

Taking the composition of these flows and letting $x_0 = (0, c, 0)^T$, we obtain

$$x=S(z)=\Psi_v^{z_1}\circ\Psi_{ad_{-f}v}^{z_2}\circ\Psi_{w_1}^{z_3}(x_0)=\begin{pmatrix} z_3(z_2+c) \\ z_2+c \\ z_1 \end{pmatrix},$$

$$z=T(x)=\begin{pmatrix} x_3 \\ x_2-c \\ x_1/x_2 \end{pmatrix},$$

see (Nijmeijer and van der Schaft, 1990, Thm. 2.36). The transformation T is a diffeomorphism on $\{x \in \mathbb{R}^3 :$

$x_2 > c\}$. Transforming (49) into a POCF, we obtain

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \underbrace{\begin{pmatrix} g - \dfrac{\beta z_3^2}{m} \\ 0 \\ \dfrac{u - R(c+z_2)z_3}{2\beta} \end{pmatrix}}_{\alpha(z_2, z_3, u)},$$

$$y = z_2.$$

We consider the second observer design described in Section 3.2. The second subsystem is

$$\dot{z}_3 = \frac{u - R(c+z_2)z_3}{2\beta} \tag{50}$$

and, since $z_2 \geq 0$, (50) has an exponentially stable equilibrium at $z_3 = 0$ when $u = 0$, and hence it satisfies Assumption A5. Although Assumption A4 is not satisfied globally, we have ensured global error convergence as the first error dynamics subsystem is LTI driven by a decaying "input",

$$\begin{pmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{pmatrix} = \begin{pmatrix} 0 & -l_1 \\ 1 & -l_2 \end{pmatrix} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} + \begin{pmatrix} \dfrac{\beta}{2m}(z_3^2 - \hat{z}_3^2) \\ 0 \end{pmatrix},$$

and hence for all $(\tilde{z}_1, \tilde{z}_2)^T(0) \in \mathbb{R}^2$, $(\tilde{z}_1, \tilde{z}_2)^T \to 0$ as $t \to \infty$.

Simulations were performed using estimated state feedback to implement state state feedback linearizing control which tracks a square wave-like reference trajectory shown in Fig. 2. The parameter values were identified from an actual physical system: $g = 9.81\,\mathrm{m/s^2}$, $\beta = 76600\,\mathrm{kg\,m^3/(s^2A^2)}$, $c = 4\,\mathrm{mm}$, $m = 0.068\,\mathrm{kg}$, $R = 11\,\Omega$. The observer eigenvalues were taken at -500 which leads to $p_0 = 2.5 \times 10^5$ and $p_1 = 1000$. The initial conditions were taken at $\tilde{x}(0) = (0.5\,\mathrm{A}, 0, 0)^T$. The corresponding estimate error trajectories are shown in Fig. 3.

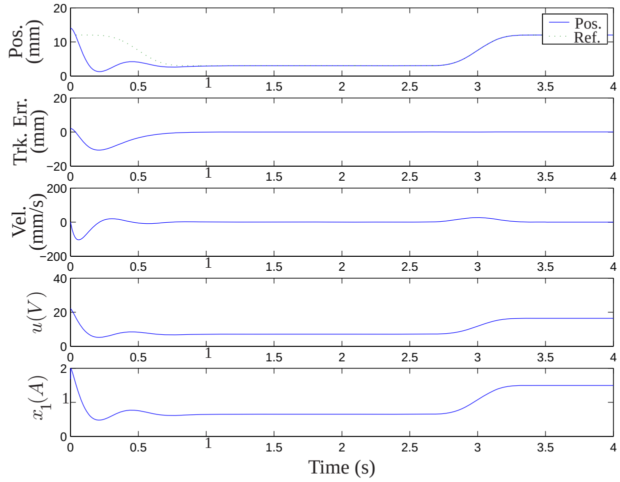

Fig. 2. Trajectories of the magnetic levitation example.

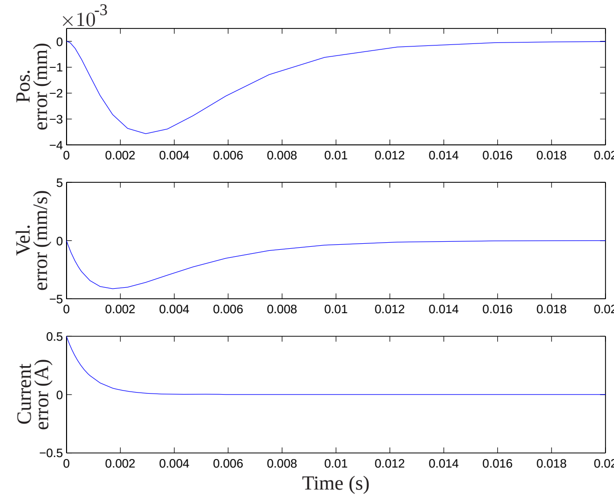

Fig. 3. Estimate errors for the magnetic levitation example.

5. Conclusion

This paper has presented two observer designs for nonlinear systems based on a new partial nonlinear observer canonical form (POCF), a detectability condition, and a Lipschitz assumption. The POCF exists under weaker conditions than the well-established OCF (Krener and Isidori, 1983) and the existing partial observer canonical forms (Jo and Seo, 2002). Two observer designs are provided. The first design has an advantage of a simple gain expression. The second design leads to a simpler error convergence proof but requires a more complicated gain formula. Two examples illustrate the design method. The synchronous generator example involves an observability matrix which is only locally nonsingular; it illustrates how the proposed design avoids the problems of inverting this matrix. This inversion is required in many canonical form designs and is not possible at points where the system is not observable. Hence, the proposed designs can admit a wide region of operation. Neither of the examples admits an OCF, as Lie bracket conditions do not hold. As the proposed approach involves weaker Lie bracket conditions, it is also more broadly applicable for this reason.

References

Amicucci G. L. and Monaco S. (1998): *On nonlinear detectability*. — J. Franklin Inst. 335B, Vol. 6, pp. 1105–1123.

Bestle D. and Zeitz M. (1983): *Canonical form observer design for non-linear time-variable systems*. — Int. J. Contr., Vol. 38, No. 2, pp. 419–431.

Birk J. and Zeitz M. (1988): *Extended Luenberger observer for non-linear multivariable systems*. — Int. J. Contr., Vol. 47, No. 6, pp. 1823–1836.

Brunovsky P. (1970): *A classification of linear controllable systems*. — Kybernetica, Vol. 6, No. 3, pp. 173–188.

Gauthier J. P., Hammouri H. and Othman S. (1992): *A simple observer for nonlinear systems — Application to bioreactors*. — IEEE Trans. Automat. Contr., Vol. 37, No. 6, pp. 875–880.

Hermann R. and Krener A. J. (1977): *Nonlinear controllability and observability*. — IEEE Trans. Automat. Contr., Vol. AC-22, No.5, pp. 728–740.

Isidori A. (1995): *Nonlinear Control Systems: An Introduction, 3-rd Edn.* — London: Springer.

Jo N. H. and Seo J. H. (2002): *Observer design for non-linear systems that are not uniformly observable*. — Int. J. Contr., Vol. 75, No. 5, pp. 369–380.

Kazantzis N. and Kravaris C. (1998): *Nonlinear observer design using Lyapunov's auxiliary theorem*. — Syst. Contr. Lett., Vol. 34, pp. 241–247.

Keller H. (1986): *Entwurf nichtlinearer Beobachter mittels Normalformen*. — Düsseldorf: VDI-Verlag.

Krener A. and Xiao M. (2002): *Nonlinear observer design in the siegel domain*. — SIAM J. Contr. Optim., Vol. 41, No. 3, pp. 932–953.

Krener A., Hubbard M., Karaham S., Phelps A. and Maag B. (1991): *Poincaré's linearization method applied to the design of nonlinear compensators*. — Proc. *Algebraic Computing in Control*, Vol. 165 of *Lecture Notes in Control and Information Science*, Berlin: Springer, pp. 76–114.

Krener A. J. and Isidori A. (1983): *Linearization by output injection and nonlinear observers*. — Syst. Contr. Lett., Vol. 3, pp. 47–52.

Krener A. J. and Respondek W. (1985): *Nonlinear observers with linearizable error dynamics*. — SIAM J. Contr. Optim., Vol. 23, No. 2, pp. 197–216.

Lynch A. and Bortoff S. (2001): *Nonlinear observers with approximately linear error dynamics: The multivariable case*. — IEEE Trans. Automat. Contr., Vol. 46, No. 7, pp. 927–932.

Marino R. and Tomei P. (1995): *Nonlinear Control Design: Geometric, Adaptive and Robust*. — London: Prentice Hall.

Moore E. H. (1920): *On the reciprocal of the general algebraic matrix*. — Bull. Amer. Math. Soc., Vol. 26, pp. 394–395.

Mukhopadhyay B. K. and Malik O. P. (1972): *Optimal control of synchronous-machine excitation by quasilinearisation techniques*. — Proc. IEE, Vol. 119, No. 1, pp. 91–98.

Nijmeijer H. and van der Schaft A. J. (1990): *Nonlinear Dynamical Control Systems*. — New York: Springer.

Phelps A. R. (1991): *On constructing nonlinear observers*. — SIAM J. Contr. Optim., Vol. 29, No. 3, pp. 516–534.

Respondek W. (1986): *Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems*. — Proc. *Algebraic and Geometric Methods in Nonlinear Control*, Dordrecht: Reidel, pp. 257–284.

Respondek W., Pogromsky A. and Nijmeijer H. (2004): *Time scaling for observer design with linearizable error dynamics*. — Automatica, Vol. 40, No. 2, pp. 277–285.

Röbenack K. and Lynch A. F. (2004): *High-gain nonlinear observer design using the observer canonical form*. — J. Franklin Institute, submitted.

Rudolph J. and Zeitz M. (1994): *A block triangular nonlinear observer normal form*. — Syst. Contr. Lett., Vol. 23, pp. 1–8.

Schweitzer G., Bleuler H. and Traxler A. (1994): *Active Magnetic Bearings*. — Zürich: VDF .

Shim H., Son Y. I. and Seo J. H. (2001): *Semi-global observer for multi-output nonlinear systems*. — Syst. Contr. Lett., Vol. 42, No. 3, pp. 233–244.

Sontag E. D. and Wang Y. (1997): *Output-to-state stability and detectability of nonlinear system*. — Syst. Contr. Lett., Vol. 29, No. 5, pp. 279–290.

Wang Y. and Lynch A. (2005): *Block triangular observer forms for nonlinear observer design*. — Automatica, (submitted).

Wang Y. and Lynch A. (2006): *A block triangular form for nonlinear observer design*. — IEEE Trans. Automat. Contr., (to appear).

Xia X.-H. and Gao W.-B. (1988): *Non-linear observer design by observer canonical form*. — Int. J. Contr., Vol. 47, No. 4, pp. 1081–1100.

Xia X. H. and Gao W. B. (1989): *Nonlinear observer design by observer error linearization*. — SIAM J. Contr. Optim., Vol. 27, No. 1, pp. 199–216.

Received: 19 April 2006
Revised: 16 June 2006