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# APPROXIMATE CONTROLLABILITY OF INFINITE DIMENSIONAL SYSTEMS OF THE $n$-th ORDER

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The objective of the article is to obtain general conditions for several types of controllability at once for an abstract differential equation of arbitrary order, instead of conditions for a fixed order equation. This innovative approach was possible owing to analyzing the $n$-th order linear system in the Frobenius form which generates a Jordan transition matrix of the Vandermonde form. We extensively used the fact that the knowledge of the inverse of a Jordan transition matrix enables us to directly verify the controllability by Chen's theorem. We used the explicit analytical form of the inverse Vandermonde matrix. This enabled us to obtain more general conditions for different types of controllability for infinite dimensional systems than the conditions existing in the literature so far. The methods introduced can be easily adapted to the analysis of other dynamic properties of the systems considered.

**Keywords:** inverse Vandermonde matrix, basic symmetrical polynomials, distributed parameter system, linear operators, controllability.

## 1. Introduction

In the literature there are many articles investigating the controllability of infinite dimensional dynamic systems of fixed order, and most often they focus on the second order (Chen and Russel, 1982; Chen and Triaggani, 1990; Huang, 1988; Ito and Kunimatsu, 1988; Respondek, 2005b; Sakawa, 1974; Sakawa, 1984; Sakawa, 1983). However, it is difficult to find works on the fourth order infinite dimensional systems, though the papers (Ito and Kunimatsu, 1991; Coleman and Wang, 1993; Kim and Renardy, 1987; Shi *et al.*, 1998; Shi *et al.*, 2001; Shubov, 1999; Xu, 2005) provide some information on the issue. In the literature there is an incomprehensible lack of papers pertaining to general $n$-th order distributed systems. So far, theorems giving conditions for controllability without constraints, with cone-type constraints, as well as absolute and relative controllability with delays in control are known for systems of arbitrary order only in the case of finite dimensional systems. The reason behind this are computational problems. One has to carry out time-consuming calculations for each order of the infinite dimensional system to find the corresponding conditions of the four types of controllability of infinite dimensional systems of any order using the classical controllability criterion (38). These calculations can be carried out in a symbolic manner only for systems of low order. Such a general approach to an equation of arbitrary order is much more sophisticated.

We found the following solutions to these problems:

- Using Chen's theorem in the examination of all four types of controllability. Chen's theorem requires only the knowledge of the inverse of a Jordan transition matrix instead of determining a block matrix.
- Bringing the $n$-th order linear system to the Frobenius form, as a Jordan matrix in this form is a Vandermonde matrix. The innovation is based on the idea of using the well-known form of the inverse of the Vandermonde matrix, which forms, in turn, a basis for controllability examination with the use of Chen's theorem.

This approach allowed us to get conditions for the examined types of controllability for infinite dimensional systems of arbitrary order.

The obtained results obviously hold true for first and second order systems with unconstrained controls, and are identical to those already presented in the literature for that case. Equivalent results for first order systems can be found in (Fattorini and Russel, 1971; Triggiani, 1976; Triggiani, 1978; Curtain and Zwart, 1995) and for second order systems in (Triggiani, 1978). Conditions for the approximate controllability of second order systems with nonnegative controls are analyzed in the paper (Respondek, 2005a) and are a particular case of Theorem 4.

It should be pointed out that the results presented in this paper can be applied only to systems whose eigenvalues and eigenvectors have explicit analytic forms. There are many systems of this type; a comprehensive work on this topic is the monograph (Butkowskij, 1979). Moreover, the test of an infinite controllability condition must be feasible by analytical means. If this is impossible, approximate methods must be involved (an example is the paper (Respondek, 2005b)).

Recent years have witnessed a few new main branches in controllability research:

- controllability of nonlinear systems (Klamka, 2000),
- stochastic controllability (Mahmudov and Zorlu, 2005),
- controllability of industrial systems (Alotaibi *et al.*, 2004; Respondek, 2007),
- numerical controllability analysis (Labbe and Trelat, 2006; Respondek, 2005b).

Besides, classical controllability is still in question (Vieru, 2005). As a possible direction for further work, we indicate stochastic controllability for systems of arbitrary order.

We start examining the controllability of the systems in question with the simplest type with neither delays nor constraints. The examination conditions for finite dimensional systems are described by Chen's theorem, which is given in Section 5. The obtained conditions of approximate controllability for any order of the infinite dimensional system are discussed in Theorem 2.

In Section 7 we examine the controllability of systems with nonnegative cone-type control constraints. For this purpose, we use Theorem 3, which is well known in the literature (Klamka, 1991; Brammer, 1972; Schmitendorf and Barmish, 1980). While examining the fourth condition of this theorem, we apply the inverse of the Vandermonde matrix. The obtained conditions for the examined infinite dimensional system are given by Theorem 4.

It is commonly known that in systems with delays in control, which we examine in Section 8, we can distinguish absolute and relative controllability. To examine these types of controllability of the infinite dimensional systems in question, it is best to use well-known theorems, i.e., Theorems 5 (Klamka, 1991, pp. 202, 130), (Klamka, 1977) and 6 (Klamka, 1976; Klamka, 1991, pp. 202,130), which we mention in Sections 8.4 and 8.5. Both theorems are based on the transformation of the initial system with delays in control into the corresponding system with no delays. This allows us to apply Chen's theorem with the use of the inverse of the Vandermonde matrix to examine the system's controllability. Consequently, we obtain a concise form of controllability conditions for the examined infinite dimensional system of arbitrary order. In order to prove it, all we needed was the widely used algebra and matrix analysis. The obtained conditions for the approximate absolute and relative controllability of the infinite dimensional system of arbitrary order are shown respectively in Theorems 7 and 8.

In Section 9, as an example, we investigate two kinds of the controllability of an elastic beam with internal damping. This example shows how to make use of the fractional powers of the state operator in the modeling of physical objects.

The apt choice of the theorems used, especially Chen's theorem, as well as the use of linear algebra allows us to concisely prove the sought conditions for the controllability types in question for the analyzed system of arbitrary order.

## 2. Problem statement

Let us consider a linear dynamic system described by the following $n$-th order abstract differential equation:

$$\begin{aligned}\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} &+ f_{n-1}(A)\frac{\mathrm{d}^{n-1}x(t)}{\mathrm{d}t^{n-1}} + \cdots + f_q(A)\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q} \\ &+ \cdots + f_1(A)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + f_0(a)x(t) \\ &= \sum_{k=0}^{M} B_k u(t-h_k), \quad t \geq t_0, \quad (1)\end{aligned}$$

where $f_q\,(A)$ denotes the following sequence of damping terms:

$$f_q\,(A) = \alpha_0^{(q)} + \alpha_1^{(q)} A + \sum_{k=2}^{\delta_q} \alpha_k^{(q)} A^{\gamma_k^{(q)}}, \qquad q = 0,1,\dots,n-1, \quad (2)$$

with initial conditions

$$\begin{aligned} x(0) &= x_0 \in D(A), \\ x^{(q)}(0) &= x_q \in X, \quad q = 1,2,\dots,n-1. \quad (3)\end{aligned}$$

Here $x(t) \in X$ ($X$ is a Hilbert space), the constant coefficients $\alpha_k^{(q)} \in \mathbb{R}$ and the exponents $\gamma_k^{(q)} \in \mathbb{R}$ of the operator $A$ are constrained by the following inequalities:

$$0 < \gamma_k^{(q)} < 1, \quad (4)$$

$k = 2,3,\dots,\delta_q$, and $q = 1,2,\dots,n-1$. The input operators $B_k$ are defined as

$$B_k u(t-h_k) = \sum_{l=1}^{p} u_l(t-h_k), \quad B_k \in L(U,X), \quad (5)$$

where

$$b_k^{(l)} \in X, \quad u_1 \in L^2_{\mathrm{loc}}([t_0,\infty),U),$$

$k = 0,1,\ldots,M$, $\quad l = 1,2,\ldots,p$, $U$ being a Hilbert control space, $\dim U = p$. The constant delays $h_k$ fulfill $0 = h_0 < h_1 < \cdots < h_k < \cdots < h_M$.

As for the state space $X$, we assume that it is a Hilbert space of square integrable functions on a bounded domain $D$, i.e., $X = L^2(D)$.

As for the state operator $A : X \supset D(A) \to X$, we assume that it

- has a domain $D(A)$ dense in $X$,
- has a compact resolvent $R(\lambda, A)$ for each $\lambda$ in the resolvent set $\rho(A)$,
- is linear,
- is generally unbounded,
- is self-adjoint,
- is positive definite,
- has eigenvectors forming a Riesz basis.

Moreover, we assume that the Frobenius matrix operator

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -f_0(A) & -f_1(A) & -f_2(A) & \cdots & -f_{n-1}(A) \end{bmatrix} \tag{6}$$

of the system (1) is an infinitesimal generator of a strongly continuous semigroup.

## 3. Transformation of the state equation

The infinite dimensional dynamic system is given by the abstract differential equation (1). Using spectral properties of the state operator $A$ (Fattorini, 1966; Fattorini, 1967; Huang, 1988; Sakawa, 1974), we can easily transform this system into the equivalent form of an infinite sequence of finite dimensional first-order linear dynamic systems with constant coefficients of the form

$$\dot{\varsigma}_i(t) = A_i\varsigma_i(t) + \sum_{k=0}^{M} B_{ki}u(t-h_k), \qquad i = 1,2,3,\ldots, \quad t \geq t_0, \tag{7}$$

where the state vector is given by

$$\varsigma_i(t) = \left[ [\varsigma'_{i1}(t)]^T \quad \cdots \quad [\varsigma'_{ij}(t)]^T \quad \cdots \quad [\varsigma'_{im_i}(t)]^T \right]^T, \qquad i = 1,2,3,\ldots, \tag{8}$$

$$\xi'_{ij} = \left[ x_{ij}(t) \ \ldots \ \frac{\mathrm{d}^q x_{ij}(t)}{\mathrm{d}t^q} \ \ldots \ \frac{\mathrm{d}^{n-1} x_{ij}(t)}{\mathrm{d}t^{n-1}} \right]^T, \qquad i = 1,2,3,\ldots, \quad j = 1,2,\ldots,m_i, \tag{9}$$

for each $i = 1,2,3,\ldots$ and $j = 1,2,\ldots,m_i$ $x_{ij}(t) = \langle x(t), \phi_{ij}\rangle_x$ denotes the $ij$-th coefficient of the Fourier sequence of the spectral representation for the element $x$ in the state space $X$, $\phi_{ij}$ is the $ij$-th eigenfunction of the state operator $A$, and $m_i$ is the multiplicity of the eigenvalues of the state operator $A$. The state matrices $A_i$ and the input matrices $B_{ki}$ are respectively the following diagonal block and block matrices:

$$A_i = \mathrm{diag}\underbrace{[A'_i|\ldots|A'_i]}_{m_i}$$

$$B_{ki} = \left[ [B'_{ki1}]^T \quad \cdots \quad [B'_{kij}]^T \quad \cdots \quad [B'_{kim_i}]^T \right]^T, \qquad k = 0,1,\ldots,M, \quad i = 1,2,3,\ldots. \tag{10}$$

The submatrices $A'_i$ in (11) are the Frobenius matrices

$$A'_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -f^*_{i0} & -f^*_{i1} & -f^*_{i2} & \cdots & -f^*_{i(n-2)} & -f^*_{i(n-1)} \end{bmatrix}, \qquad i = 1,2,3,\ldots \tag{11}$$

The submatrices $B'_{kij}$ are given by (12) for $i = 1,2,3,\ldots$ and $j = 1,2,\ldots,m_i$ and $k = 0,1,\ldots,M$. Based on (2), the constant coefficients $f^*_{iq}$ in the submatrices $A'_i$ are defined by

$$f^*_{iq} = \alpha_0^{(q)} + \alpha_1^{(q)}\lambda_i + \sum_{k=2}^{\delta_q} \alpha_k^{(q)} \lambda_i^{\gamma_k^{(q)}}, \tag{13}$$

where $q = 0,1,\ldots,n-1$, and for each $i = 1,2,3,\ldots$, $\lambda_i$ is the $i$-th eigenvalue of the state operator $A$.

The state matrix of the system (7) is a block diagonal matrix (cf. (10)). The determinant of this matrix is a product of submatrix determinants (Kaczorek, 1998, pp. 70). Thus the characteristic equation of (7) follows directly from (14):

$$\Big( s_i^n + f^*_{i(n-1)} s_i^{n-1} + \cdots + f^*_{iq} s_i^q + \ldots + f^*_{i1} s_i + f^*_{i0} \Big)^{m_i} = 0, \qquad i = 1,2,3,\ldots. \tag{14}$$

We assumed that the Frobenius matrix operator of the system (1) is an infinitesimal generator of a strongly continuous semigroup. Thus, all the real parts of the roots of the characteristic equation (14) have an upper limit, i.e.,

$$\lim_{i\to\infty} \mathrm{Re}\,[s_{ik}] < \infty, \quad i = 1,2,3,\ldots, \quad k = 1,2,\ldots,r_i. \tag{15}$$

$$B'_{kij} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \langle b_k^{(1)}, \psi_{ij} \rangle_X & \cdots & \langle b_k^{(l)}, \psi_{ij} \rangle_X & \cdots & \langle b_k^{(p)}, \psi_{ij} \rangle_X \end{bmatrix}. \tag{12}$$

It is well known that the approximate controllability of the system (1) is equivalent to the approximate controllability of the infinite sequence of the finite dimensional systems (7). This fact enables us to apply Theorems 1, 3, 5 and 6, concerning different types of controllability of finite dimensional systems, to each finite dimensional subsystem in the sequence (7). This methodology is used in the analysis of all the types of controllability investigated in this paper, i.e., in Theorems 2, 4, 7 and 8.

## 4. Jordan decomposition of the state matrix

This decomposition is convenient while verifying all types of controllability investigated in this paper. It can be easily noticed that this matrix has $n$ distinct eigenvalues, each with the same multiplicity $m_i$ (16), equal to the root $s_{iq}$ of the characteristic equation (14):

$$\sigma(A_i) = \{s_{i1}, \dots, s_{iq}, \dots, s_{in}\}, \quad i = 1, 2, 3, \dots . \tag{16}$$

Using these eigenvalues, we can prove that the Jordan canonical form of the state matrices $A_i$ (10) has the form of the following diagonal matrix:

$$J(A_i) = \operatorname{diag}\Big[ \underbrace{s_{i1} \dots s_{i1}}_{m_i \text{ times}} \cdots \underbrace{s_{iq} \dots s_{iq}}_{m_i \text{ times}} \cdots \underbrace{s_{in} \dots s_{in}}_{m_i \text{ times}} \Big], \quad i = 1, 2, 3, \dots . \tag{17}$$

The transformation matrix $T(A_i)$ has a rather sophisticated form of the block matrix

$$T(A_i) = \begin{bmatrix} T_{i1} & \cdots & T_{iq} & \cdots & T_{in} \end{bmatrix}, \tag{18}$$

where

$$T_{iq} = \underbrace{\begin{bmatrix} 0 & \cdots & t_{iq} \\ \vdots & \iddots & \vdots \\ t_{iq} & \cdots & 0 \end{bmatrix}}_{m_i \text{ times}}, \tag{19}$$

$q = 1, 2, \dots, n$, and $i = 1, 2, 3, \dots$ .

Note that 0 in (19) denotes the $n$-element vertical zero vector. It is well known (Górecki, 1986, pp. 86) that the Jordan canonical form of the Frobenius matrix (11) is a Vandermonde matrix, and therefore the blocks $t_{iq}$ in the block Jordan transformation matrix (19) are vertical vectors defined by

$$[t_{i1} \,|\, \dots \,|\, t_{iq} \,|\, \dots \,|\, t_{in}] = \begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 \\ s_{i1} & \cdots & s_{iq} & \cdots & s_{in} \\ s_{i1}^2 & \cdots & s_{iq}^2 & \cdots & s_{in}^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{i1}^{n-1} & \cdots & s_{iq}^{n-1} & \cdots & s_{in}^{n-1} \end{bmatrix}, \quad i = 1, 2, 3, \dots . \tag{20}$$

The basis for the entire controllability research in this paper is Chen's controllability Theorem 1. In that theorem the inverse of the Jordan transformation matrix $T(A_i)$ plays a crucial role. This inverse is also useful in the verification of controllability with cone-type control constraints, in the formula expressing the eigenvectors of the transposed state matrix. It is well known (Górecki, 1986, pp. 86) that the inverse of the Vandermonde matrix is expressed by the so-called basic symmetric polynomials. Further in this section, we shall present how the inverse is built in the case of the block transformation matrix $T(A_i)$ (19). Namely, the desired inverse $T^{-1}(A_i)$ can be expressed by

$$T^{-1}(A_i) = \begin{bmatrix} (T_{i1}^o)^T & \cdots & (T_{iq}^o)^T & \cdots & (T_{in}^o)^T \end{bmatrix}^T, \tag{21}$$

where

$$T_{iq}^o = \underbrace{\begin{bmatrix} 0 & \cdots & t_{iq}^o \\ \vdots & \iddots & \vdots \\ t_{iq}^o & \cdots & 0 \end{bmatrix}}_{m_i \text{ times}}, \tag{22}$$

$q = 1, 2, \dots, n$, and $i = 1, 2, 3, \dots$ .

Note that 0 in (22) denotes the $n$-element horizontal zero vector and the blocks $t_{iq}^o$ are now horizontal vectors defined by (23) (Górecki, 1986, pp. 86). Here $w_{ik}^{(v)}$ denotes for $k > 0$ the $k$-th order basic symmetric polynomial in $n-1$ variables $s_{i1}, s_{i2}, \dots, s_{i(v-1)}, s_{i(v+1)}, \dots, s_{in}$, and $w_{i0}^{(v)} \overset{\text{df}}{=} 1$, i.e., for each $i = 1, 2, 3, \dots$ we have (24).

$$\begin{bmatrix} t_{i1}^{o} \\ \vdots \\ t_{in}^{o} \end{bmatrix} = \begin{bmatrix} \dfrac{w_{i(n-1)}^{(1)}(-1)^{1-1}}{(s_{i2}-s_{i1})\dots(s_{in}-s_{i1})} & \dfrac{w_{i(n-2)}^{(1)}(-1)^{2-1}}{(s_{i2}-s_{i1})\dots(s_{in}-s_{i1})} & \cdots & \dfrac{w_{i0}^{(1)}(-1)^{n-1}}{(s_{i2}-s_{i1})\dots(s_{in}-s_{i1})} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{w_{i(n-1)}^{(n)}(-1)^{1-1}}{(s_{i1}-s_{in})\dots\left(s_{i(n-1)}-s_{in}\right)} & \dfrac{w_{i(n-2)}^{(n)}(-1)^{2-1}}{(s_{i1}-s_{in})\dots\left(s_{i(n-1)}-s_{in}\right)} & \cdots & \dfrac{w_{i0}^{(n)}(-1)^{n-1}}{(s_{i1}-s_{in})\dots\left(s_{i(n-1)}-s_{in}\right)} \end{bmatrix},$$

$$i = 1, 2, 3, \dots \tag{23}$$

$$\begin{cases} w_{i1}^{(v)} &= s_{i1} + s_{i2} + \dots + s_{i(v-1)} + s_{i(v+1)} + \dots + s_{in}, \\ w_{i2}^{(v)} &= s_{i1}s_{i2} + \dots + s_{i1}s_{i(v-1)} + s_{i1}s_{i(v+1)} + \dots + s_{i1}s_{in} + \dots + s_{i(n-1)}s_{in}, \\ \vdots & \\ w_{i(n-1)}^{(v)} &= s_{i1}s_{i2}\dots s_{i(v-1)}s_{i(v+1)}\dots s_{in}. \end{cases} \tag{24}$$

Summarising this section, the Jordan transformation matrix $T(A_i)$ of the infinite sequence of the finite dimensional systems (7) is given by (18)–(20), and its inverse $T^{-1}(A_i)$ is given by (21)–(24).

## 5. Chen's controllability theorem

Consider a linear, stationary, finite dimensional dynamic system described by

$$x(t) = A_0 x(t) + B_0 u(t), \quad t \geq 0, \tag{25}$$

where $A_0$ and $B_0$ are constant matrices with dimensions $n \times n$ and $n \times p$, respectively. Chen's controllability theorem pertains to the system (25) in the Jordan canonical form. Hence, before formulating Chen's theorem, the following remark will be useful:

**Remark 1.** (Klamka, 1991; Chen, 1970, pp. 22) The controllability of the dynamic system (25) is invariant under any linear transformation $x = Tz$, where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $T$ is an $n \times n$-dimensional, nonsingular transformation matrix.

Assume that the Jordan canonical form of the dynamic system (25) is represented by the matrices $J$ and $G = T^{-1}B_0$, where

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_k \end{bmatrix}, \tag{26}$$

$$J_i = \begin{bmatrix} J_{i1} & & 0 \\ & \ddots & \\ 0 & & J_{ir(i)} \end{bmatrix}, \quad G_i = \begin{bmatrix} G_{i1} \\ G_{i2} \\ \vdots \\ G_{ir(i)} \end{bmatrix},$$
$$i = 1, 2, \dots, k, \tag{27}$$

$$J_{ij} = \begin{bmatrix} s_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & s_i & 1 \\ 0 & & & s_i \end{bmatrix}, \quad G_{ij} = \begin{bmatrix} g_{ij1} \\ g_{ij2} \\ \vdots \\ g_{ijn_{(ij)}} \end{bmatrix},$$
$$i = 1, 2, \dots, k, \quad j = 1, 2, \dots, r\,(i). \tag{28}$$

Here $s_1, s_2, \dots, s_k$ are distinct eigenvalues of the matrix $A_0$ with multiplicities $n_i$, $i = 1, 2, \dots, k$; $J_i$, $i = 1, 2, \dots, k$ are $n_i \times n_i$-dimensional matrices containing all the Jordan blocks associated with the eigenvalues $s_i$; $J_{ij}$, $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, r\,(i)$ are $n_{ij} \times n_{ij}$-dimensional Jordan blocks in $J_i$; $r(i)$ is the number of Jordan blocks in the submatrix $J_i$, $i = 1, 2, \dots, k$; $G_i$, $i = 1, 2, \dots, k$ are $n_i \times m$-dimensional submatrices of the matrix $G$ corresponding to the submatrices $J_i$; $G_{ij}$, $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, r\,(i)$ are $n_{ij} \times m$-dimensional submatrices of the matrix $G_i$ corresponding to the Jordan blocks $J_{ij}$; $g_{ijn_{(ij)}}$, $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, r\,(i)$ are the rows of the submatrix $G_{ij}$ corresponding to the rows of the Jordan blocks $J_{ij}$.

Now, using the Jordan canonical form of the dynamic system (25) represented by the matrices (26)–(28), we can recall Chen's controllability theorem.

**Theorem 1.** *(Chen, 1970; Klamka, 1991, pp. 25) The dynamic system (25) is controllable if and only if for each $i = 1, 2, \dots, k$ the rows $g_{i1n_{i1}}, g_{i2n_{i2}}, \dots, g_{ir(i)n_{ir(i)}}$ of* *the matrix G are linearly independent over the field of the complex numbers.*

## 6. Unconstrained approximate controllability without delays

In this section we will assume no delays, so $M = 0$ and the control space $U = \mathbb{R}^p$. Necessary and sufficient conditions for the approximate controllability of the analysed dynamic system (1) will be formulated and proved.

The condition for this kind of controllability of first-order systems is presented in the book (Curtain and Zwart, 1995, pp. 163), cf. Theorem 4.2.1. In this section we shall generalize this result to dynamic systems of arbitrary order. First, recall the definition of approximate controllability.

**Definition 1.** (Klamka, 1991, pp. 2, 130) The dynamic system (1) is *approximately controllable* if and only if there exists a control $u(t)$ which will transfer the system from any given initial state $x_0 \in \tilde{X}$ to any final state $x_1 \in \tilde{X}$ in a finite time, where $\tilde{X}$ is a dense subspace of $X$.

In order to verify the controllability, we shall make use of the Jordan canonical form of the state equation (7). According to Remark 1, the controllability of a linear, stationary finite dimensional system is invariant under any nonsingular linear transformation. Thus, let us transform the sequence of the dynamic systems (7) using the Jordan transition matrix $T(A_i)$ (19):

$$\varsigma_i(t) = T(A_i)\theta_i(t), \quad i = 1, 2, 3, \ldots. \tag{29}$$

After the linear transformation (29), Eqn. (7) gets its Jordan canonical form

$$\dot{\theta}_i(t) = J(A_i)\,\theta_i(t) + G_i u(t), \quad i = 1, 2, 3, \ldots, \tag{30}$$

where $J(A_i)$ is given by (17) and $G_i = T_i^{-1}(A_i)B_{0i}$. The input matrix $B_{0i}$ is given by (10). Now, let us determine $G_i = T_i^{-1}(A_i)B_{0i}$ on the basis of (10) and (21)–(24). We obtain (31), where $\otimes$ denotes the Kronecker product (Bellman, 1960, pp. 255) and the matrices $(B_{0i}^*)'$ can be obtained from (32) by reversing the order of the rows and setting $k = 0$.

Now let us return to the verification of the controllability of the dynamic system (7) in the form (30). The conditions for the controllability of linear dynamic systems in the canonical Jordan form are given by Theorem 1. We can obtain a sequence of conditions (33) by applying Chen's theorem to the system (30) with respect to the particular Jordan canonical form of the system (30) and the matrix $G_i = T_i^{-1}(A_i)B_{0i}$. The sequence of conditions (33) is the following:

$$\operatorname{rank}\left[\frac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}(s_{ir}-s_{iq})}(B_{0i}^*)'\right] = m_i,$$
$$q = 1, 2, \ldots, n, \quad i = 1, 2, 3, \ldots. \tag{33}$$

After simple linear transformations, Eqns. (33) can be rewritten in the most compact form of one equation:

$$\operatorname{rank}[B_{0i}^*] = m_i, \quad i = 1, 2, 3, \ldots. \tag{34}$$

**Theorem 2.** *The dynamic system (1) without delays in control ($M = 0$) is approximately controllable if and only if the infinite sequence of the equalities (34) is fulfilled.*

**Corollary 1.** *The dynamic system (1) without delays in control ($M = 0$), with only single multiplicities of the eigenvalues of the state operator $A$, is approximately controllable if and only if the infinite sequence of the equalities*

$$\sum_{k=1}^{p} \langle b_0^{(k)}, \phi_{i1}\rangle_X^2 \neq 0, \quad i = 1, 2, 3, \ldots \tag{35}$$

*is fulfilled.*

## 7. Approximate controllability without delays with nonnegative cone-type constraints

In this section we shall also assume no delays, so $M = 0$, but as the control space we shall take a nonnegative cone $U \subset \mathbb{R}_+^p$. The necessary and sufficient conditions for the so-called $U$-controllability with this type of constraints of the analysed dynamic system (1) will be formulated and proved as Theorem 4.

**Definition 2.** (Klamka, 1991, pp. 36, 130) The dynamic system (1) is *globally approximately $U$-controllable to zero* if for each initial state $x_0 \in \tilde{X}$, where $\tilde{X}$ is a dense subspace of $X$, there exists an admissible control $u \in L^2([t_0, \infty), U)$ such that the corresponding trajectory $x(t, x(t_0), u)$ of the dynamic system satisfies

$$x(t_1, x(t_0), u) = 0 \tag{36}$$

for some $t_1 \in [t_0, \infty)$. The conditions for the approximate $U$-controllability of finite dimensional linear dynamic systems with cone-type control constraints are well known and presented in (Klamka, 1991, pp. 52), cf. Theorem 1.9.1, and (Brammer, 1972; Schmitendorf and Barmish, 1980).

**Theorem 3.** *(Brammer, 1972; Klamka, 1991; Schmitendorf and Barmish, 1980) The dynamic system*

$$\dot{x}(t) = A_0 x(t) + B_0 u(t), \quad t \geq 0,$$

$$G_i = \begin{bmatrix} \dfrac{w_{i0}^{(1)}(-1)^{n-1}}{\prod\limits_{r=2}^{n}(s_{ir}-s_{i1})} & \cdots & \dfrac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}(s_{ir}-s_{iq})} & \cdots & \dfrac{w_{i0}^{(n)}(-1)^{n-1}}{\prod\limits_{r=1}^{n-1}(s_{ir}-s_{in})} \end{bmatrix}^T \otimes \left[(B_{0i}^*)'\right], \quad i=1,2,3,\ldots, \tag{31}$$

$$B_{ki}^* = \begin{bmatrix} \langle b_k^{(1)},\phi_{i1}\rangle_X & \ldots & \langle b_k^{(l_2)},\phi_{i1}\rangle_X & \ldots & \langle b_k^{(p)},\phi_{i1}\rangle_X \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle b_k^{(1)},\phi_{il_1}\rangle_X & \ldots & \langle b_k^{(l_2)},\phi_{il_1}\rangle_X & \ldots & \langle b_k^{(p)},\phi_{il_1}\rangle_X \\ \langle b_k^{(1)},\phi_{im_i}\rangle_X & \ldots & \langle b_k^{(l_2)},\phi_{im_i}\rangle_X & \ldots & \langle b_k^{(p)},\phi_{im_i}\rangle_X \end{bmatrix},$$

$$k=0,1,\ldots,M, \quad \text{and} \quad i=1,2,3,\ldots \tag{32}$$

*where $A_0$ and $B_0$ are constant matrices with dimensions $n\times n$ and $n\times p$, respectively, is globally $U$-controllable to zero if and only if the following conditions are simultaneously satisfied:*

*(i) There exists a $w\in U$ such that $B_0w=0$.*

*(ii) The convex hull $CH(U)$ has a nonempty interior in the space $\mathbb{R}^p$.*

*(iii) There holds*

$$\operatorname{rank}\left[B_0|A_0B_0|A_0^2B_0|\ldots|A_0^{n-1}B_0\right]=n. \tag{37}$$

*(iv) There is no real eigenvector $v\in\mathbb{R}^n$ of the matrix $A_0^T$ satisfying $v^TB_0w\le 0$ for all $w\in U$.*

*(v) No eigenvalue of the matrix $A_0$ has a positive real part.*

**Theorem 4.** *Assume that all the assumptions made in Sections 1–4 are valid. The dynamic system (1) is globally approximately $U$-controllable to zero with nonnegative cone-type controls if and only if the following conditions are simultaneously satisfied:*

*(i) There exists a $w_i\in U$ such that $B_{0i}^*w_i=0$ for each $i=1,2,3,\ldots$.*

*(ii) The convex hull $CH(U)$ has a nonempty interior in the space $\mathbb{R}_p$.*

*(iii) The infinite sequence of the equalities (35) are fulfilled.*

*(iv) For every $i$ in the set*

$$\{i\in\mathbb{Z}_+ : \exists\, q\in\{1,2,\ldots,n\}, \operatorname{Im}[s_{iq}]=0\}$$

*in each $n$-th row of the $i$-th input matrix $B_{0i}$ in (11) there must exist a pair of the scalar products of the opposite sign.*

*(v) No eigenvalue of the sequence of the state matrices (11) and (12) has a positive real part.*

*Proof.* As was described in Section 3, the approximate controllability of the system (1) is equivalent to the approximate controllability of the infinite sequence of the finite dimensional systems (7). Moreover, it is well known that any finite dimensional subsystem (7) is $U$-controllable if and only if Conditions (i–v) of Theorem 3 are satisfied simultaneously. Therefore, the original system (1) is approximately $U$-controllable if and only if Conditions (i)–(v) of Theorem 3 are satisfied for each of the finite dimensional subsystems in the infinite sequence (8). Now, let us apply sequentially each of the five conditions of Theorem 3 to every subsystem from the infinite sequence (8). Conditions (i), (ii) and (v) of this theorem follow directly from Theorem 3

Condition (iii) is that of unconstrained controllability and was given in this paper for the system (1) by Theorem 2.

Condition (iv) pertains to the real eigenvectors of the state matrices $A_i$ in (10) and (11) corresponding to the real eigenvalues of the characteristic equation (14). Write

$$\mathbb{Z}_{\mathrm{Re}} = \Big\{(i\in\mathbb{Z}_+, q\in\{1,2,\ldots,n\}) : \operatorname{Im}[s_{iq}]=0\Big\}. \tag{38}$$

As has already been mentioned, the eigenvectors of the state matrices $A_i$ in (10) and (11) have the form of the Vandermonde block matrix (18)–(20). The eigenvectors of the transposed state matrix are

$$T(A_i^T) = \left[T^{-1}(A_i)\right]^T, \quad i=1,2,3,\ldots. \tag{39}$$

On the basis of (39) and (21)–(23), the eigenvectors $v_i^{(l)}(s_{iq})$ of the matrices $A_i^T$ are given by (40).

Let us determine the term $B_{0i}w_i$. Using (10) and (12), we get (41).

$$v_i^{(l)}\left(s_{iq}\right) = \Bigg[\underbrace{0 \cdots 0}_{n\times m_i - n\times l \text{ times}} \quad \frac{w_{i(n-1)}^{(q)}(-1)^{1-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}\left(s_{ir}-s_{iq}\right)} \quad \cdots \quad \frac{w_{i(n-2)}^{(q)}(-1)^{2-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}\left(s_{ir}-s_{iq}\right)} \quad \cdots \quad \frac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}\left(s_{ir}-s_{iq}\right)} \quad \underbrace{0 \quad \cdots \quad 0}_{n\times l - n \text{ times}}\Bigg]^T,$$

$$(i,q)\in \mathbb{Z}_{\mathrm{Re}}. \tag{40}$$

$$B_{0i}w_i = \Bigg[\underbrace{0 \quad \cdots \quad 0}_{n-1} \sum_{k=1}^{p}\langle b_0^{(k)},\phi_{i1}\rangle_X u_k \cdots \underbrace{0 \quad \cdots \quad 0}_{n-1} \sum_{k=1}^{p}\langle b_0^{(k)},\phi_{il}\rangle_X u_k \cdots \underbrace{0 \quad \cdots \quad 0}_{n-1} \sum_{k=1}^{p}\langle b_0^{(k)},\phi_{im_i}\rangle_X u_k\Bigg]^T. \tag{41}$$

Finally, let us evaluate the term $\left(v_i^{(l)}\left(s_{iq}\right)\right)^T B_{0i}w_i$. Combining (40) with (41), we get

$$\left(v_i^{(l)}\left(s_{iq}\right)\right)^T B_{0i}w_i = y\frac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}\left(s_{ir}-s_{iq}\right)}\sum_{k=1}^{p}\langle b_0^{(k)},\phi_{il}\rangle_X u_k,$$
$$(i,q)\in\mathbb{Z}_{\mathrm{Re}},\quad l=1,2,\dots,m_i. \tag{42}$$

Since the controls are constrained to a nonnegative cone, by the particular form (42) of the term $(v_i^{(l)}\left(s_{iq}\right))^T B_{0i}w_i$ for the analysed dynamic system (7), we deduce that Condition (iv) reduces to the requirement that the expression $(v_i^{(l)}(s_{iq}))^T B_{0i}w_i,\ (i,q)\in\mathbb{Z}_{\mathrm{Re}},\ \ l=1,2,\dots,m_i$ , given by (42), have values of both signs in the admissible control space. Under this condition there is no eigenvector $v_i^{(l)}$ of the matrix $A_i^T$ such that (43) holds, i.e.,

$$\forall (i,q)\in\mathbb{Z}_{\mathrm{Re}}\,\forall w_i\in U\ \ \forall l=1,2,\dots,m_i,$$
$$(v_i^{(l)}(s_{iq}))^T \quad B_{0i}w_i\le 0. \tag{43}$$

From (42) it can be deduced that the expression $\left(v_i^{(l)}(s_{iq})\right)^T B_{0i}w_i$ will have values of both signs for the nonnegative controls if and only if in each $n$-th row in the matrix (10) there exists a pair of scalar products of opposite signs, for each $i$ in the set $\mathbb{Z}_{\mathrm{Re}}$. ■

## 8. Unconstrained approximate controllability with delays

In this section we shall assume delays in control and unconstrained controls, so that $U=\mathbb{R}^p$. To pursue the objective of analysing the approximate controllability of the infinite dimensional system with delays (1), let us present this notion first. For the dynamic system of the form (1), besides the instantaneous state $x(t)\in X$, we also introduce the notion of the so-called *complete state* at time $t$, $z(t)=\{x(t),u_t(s)\}$, where $u_t(s)=u(s)$ for $s\in[t-h_M,t]$ (Klamka, 1991, pp. 195). Therefore we distinguish two basic notions of approximate controllability for the dynamic system (1), namely: relative approximate controllability and absolute approximate controllability (Klamka, 1991, pp. 195, 130). Definitions 3 and 4 are taken from (Klamka, 1991, pp. 195, 130) and adapted to the dynamic system (1), i.e., with multiple, lumped time-invariant delays in control.

**Definition 3.** (Klamka, 1977; Klamka, 1991, pp. 130, 195). The dynamic system (1) is *absolutely approximately controllable in* $[t_0,t_1]$ if for any initial complete state

$$z(t_0)=\{x(t_0),u_t(s)\},\ x(t_0)\in\tilde{X},$$

any state $x_1\in\tilde{X}$ , where $\tilde{X}$ is a dense subspace of $X$, and an arbitrary function $w\in L^2\left([t_1-h_M,t_1],U\right)$, there exists a control $u\in L^2\left([t_0,t_1],U\right)$ such that the complete state of the dynamic system (1) satisfies

$$z(t_1)=\{x_1,w\}. \tag{44}$$

**Definition 4.** (Klamka, 1976; Klamka, 1991, pp. 130, 195). The dynamic system (1) is *relatively approximately controllable in* $[t_0,t_1]$ if for any initial complete state $z(t_0)=\{x(t_0),u_t(s)\},\ x(t_0)\in\tilde{X}$, any state $x_1\in\tilde{X}$, where $\tilde{X}$ is a dense subspace of $X$, there exists a control $u\in L^2\left([t_0,t_1],U\right)$ such that the corresponding trajectory $x\left(t,z(t_0),u\right)$ of the system (1) satisfies

$$x\left(t_1,z(t_0),u\right)=x_1. \tag{45}$$

Definition 3 immediately implies that absolute controllability has sense only for a sufficiently long time horizon, i.e., when $t_1 > t_0 + h_M$ (Klamka, 1991). There are some known theorems for verifying the relative and absolute controllability of linear time-varying systems with delays and control. Let us present two main theorems in the form adapted to the stationary dynamic system (1).

**Theorem 5.** *(Klamka, 1977; Klamka, 1991, pp. 207, 130). The dynamic system (1) is absolutely approximately controllable in $[t_0, t_1]$, $t_1 > t_0 + h_M$, if and only if the dynamic system without delays in control*

$$\zeta(t) = A_i\zeta_i(t)+\hat{B}_i u(t), \quad t \in [t_0, t_1], \quad t_1 > t_0+h_M, \quad i = 1, 2, 3, \dots, \tag{46}$$

*where*

$$\hat{B}_i = \sum_{k=0}^{M} e^{-A_i h_k} B_{ki}, \quad i = 1, 2, 3, \dots, \tag{47}$$

*is approximately controllable in $[t_0, t_1 - h_M]$.*

To simplify the notation, with no loss of generality, we may assume that there exists an index $k_0 \le M$ such that $t_1 - h_{k_0} = 0$. If such $k_0$ does not exist, then we introduce an additional delay $h_{k_0}$ with the control matrix $B_{k_0 i} = 0$ (Klamka, 1991). The index $k_0$ plays an important role in the definition of relative controllability. Relative controllability is defined for an arbitrary time interval $[t_0, t_1]$, $t_1 > t_0$ (Klamka, 1991).

**Theorem 6.** *(Klamka, 1976; Klamka, 1991, pp. 202, 130). The dynamic system (1) is relatively approximately controllable in $[t_0, t_1]$, for any time interval $t_1 > t_0$, if and only if the dynamic system without delays in control*

$$\zeta_i(t) = A_i\zeta_i(t) + \tilde{B}_i u(t), \quad t \in [t_0, t_1], \quad t_1 > t_0, \quad i = 1, 2, 3, \dots, \tag{48}$$

*where*

$$\tilde{B}_i = [B_{0i}|B_{1i}|\dots|B_{(k_0-1)i}], \quad t \in [t_0, t_1], \quad t_1 > t_0, \quad w \in \mathbb{R}^{k_0 p}, \quad i = 1, 2, 3, \dots, \tag{49}$$

*is approximately controllable in $[t_0 + h_{k_0-1}, t_1]$.*

**Theorem 7.** *The dynamic system (1) is approximately absolutely controllable in the time interval $[t_0, t_1]$, $t_1 > t_0 + h_M$, if and only if the infinite sequence of equalities*

$$\operatorname{rank} \sum_{k=0}^{M} e^{-s_{iq}h_k} B^*_{ki} = m_i, \quad q = 1, 2, \dots, n, \quad i = 1, 2, 3, \dots, \tag{50}$$

*is fulfilled, where $B^*_{ki}$ is given by (33), $s_{iq}$ are the eigenvalues (16) of the state operator $A_i$ in (11), $h_k$ are the delays, and $m_i$ are the eigenvalue multiplicities of the state operator (6).*

*Proof.* We shall prove the conditions for the absolute controllability of the system (1) in the form of the sequence (8) by Theorems 1–4. First, let us determine the matrix $\hat{B}_i$ for the system (8) (the matrix (47) from Theorem 4):

$$\hat{B}_i = \sum_{k=0}^{M} e^{-A_i h_k} B_{ki} = \sum_{k=0}^{M} T_i e^{-J_i h_k} T_i^{-1} B_{ki}, \quad i = 1, 2, 3, \dots. \tag{51}$$

The term $T_i^{-1}\hat{B}_i$ plays a key role in Chen's theorem. From (10), (17), (21) and (51) we have (52), where $J_{m_i}(f)$ is a diagonal matrix with $m_i$ diagonal entries equal to $f$. Observing that only every $n$-th row in the sequence of the matrices $B_{ki}$ (10) is nonzero, from (52) and (21)–(23) we directly get (53), where the matrices $(B^*_{ki})'$ can also be obtained from $B^*_{ki}$, given by (32), by reversing the order of the rows. Now, let us return to the verification of the controllability of the dynamic system (1) in the form (46) from Theorem 4. Applying Chen's theorem to (53), the system considered, presented in the corresponding form without delays with the input matrix $\hat{B}_i$ (given by (51)), is approximately controllable if and only if

$$\operatorname{rank} \left[ \frac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n} (s_{ir} - s_{iq})} \sum_{k=0}^{M} e^{-s_{iq}h_k} (B^*_{ki})' \right] = m_i, \quad q = 1, 2, \dots, n, \quad i = 1, 2, 3, \dots. \tag{54}$$

Applying the basic linear algebra rules to (54), we get (50). Chen's theorem gives the controllability conditions at any time interval, including obviously the interval $[t_0, t_1 - h_M]$ required by Theorem 4. ■

**Theorem 8.** *The dynamic system (1) is approximately relatively controllable in $[t_0, t_1]$, for any time interval $t_1 > t_0$, if and only if*

$$\operatorname{rank} \left[ B^*_{0i} \,\middle|\, B^*_{1i} \,\middle|\, \dots \,\middle|\, B^*_{(k_0-1)\,i} \right] = m_i, \quad i = 1, 2, 3, \dots, \tag{55}$$

*where $B^*_{ki}$ is given by (32).*

*Proof.* The proof is based on Theorems 1 and 5. The term $T_i^{-1}\tilde{B}_i$ from Theorem 5 can be easily calculated as (56), using (21)–(23) and (49), where the matrices $(B^*_{ki})'$ can be obtained from the input matrices $B^*_{ki}$ (32) by reversing

$$
\begin{aligned}
T_i^{-1}\hat{B}_i &= \sum_{k=0}^{M} e^{-J_i h_k} T_i^{-1} B_{ki} \\
&= \sum_{k=0}^{M} \operatorname{diag}\left[ J_{m_i}\left(e^{-s_{i1}h_k}\right), \ \dots\ , J_{m_i}\left(e^{-s_{iq}h_k}\right), \ \dots\ , J_{m_i}\left(e^{-s_{in}h_k}\right) \right] \\
&\quad \times \left[ \begin{array}{ccccc} \left(T_{i1}^{o}\right)^T & \cdots & \left(T_{iq}^{o}\right)^T & \cdots & \left(T_{in}^{o}\right)^T \end{array} \right]^T B_{ki} \\
&= \sum_{k=0}^{M} \left[ \begin{array}{ccccc} e^{-s_{i1}h_k}\left(T_{i1}^{o}\right)^T & \cdots & e^{-s_{iq}h_k}\left(T_{iq}^{o}\right)^T & \cdots & e^{-s_{in}h_k}\left(T_{in}^{o}\right)^T \end{array} \right]^T B_{ki}, \quad i = 1,2,3,\dots \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
T_i^{-1}\hat{B}_i = \sum_{k=0}^{M} & \left[ \begin{array}{ccccc} \dfrac{w_{i0}^{(1)}(-1)^{n-1}}{\prod\limits_{r=2}^{n}(s_{ir}-s_{i1})} e^{-s_{i1}h_k} & \cdots & \dfrac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}(s_{ir}-s_{iq})} e^{-s_{iq}h_k} & \cdots & \dfrac{w_{i0}^{(n)}(-1)^{n-1}}{\prod\limits_{r=1}^{n-1}(s_{ir}-s_{in})} e^{-s_{in}h_k} \end{array} \right]^T \\
& \otimes \left[ (B_{ki}^{*})' \right], \qquad i = 1,2,3,\dots \qquad (53)
\end{aligned}
$$

$$
\begin{aligned}
T_i^{-1}\tilde{B}_i = & \left[ \begin{array}{ccccc} \dfrac{w_{i0}^{(1)}(-1)^{n-1}}{\prod\limits_{r=2}^{n}(s_{ir}-s_{i1})} & \cdots & \dfrac{w_{i0}^{(q)}(-1)^{n-1}}{\prod\limits_{\substack{r=1\\ r\neq q}}^{n}(s_{ir}-s_{iq})} & \cdots & \dfrac{w_{i0}^{(n)}(-1)^{n-1}}{\prod\limits_{r=1}^{n-1}(s_{ir}-s_{in})} \end{array} \right]^T, \\
& \otimes \left[ (B_{0i}^{*})' \,\middle|\, (B_{1i}^{*})' \,\middle|\, \dots \,\middle|\, \left(B_{(k_0-1)i}^{*}\right)' \right], \qquad i = 1,2,3,\dots \qquad (56)
\end{aligned}
$$

the order of the rows. Applying Chen's theorem to the term $T_i^{-1}\tilde{B}_i$, (56) leads to the sequence of equalities

$$
\operatorname{rank}\left[ B_{0i}^{*} \,\middle|\, B_{1i}^{*} \,\middle|\, \dots \,\middle|\, B_{(k_0-1)\,i}^{*} \right] = m_i, \qquad i = 1,2,3,\dots . \qquad (57)
$$

As as previously been mentioned, Chen's theorem gives the controllability condition on any time interval, in this case also including the interval $[t_0 + h_{k_0-1}, t_1]$ required by Theorem 5. Thus (58) fulfills the conditions of Theorem 5 for any $k_0 \leq M$ . ∎

## 9. Example

We shall show how to apply some results obtained in this paper to the investigation of the controllability of an elastic beam with internal damping and two control forces. First, we shall show how to transform a classical, distributed parameter mathematical model into an abstract differential equation. Next, we shall transform it into the form of an infinite sequence of finite dimensional equations. Finally, we shall analyse two types of controllability of the elastic beam considered. The example shows the importance of the damping terms in the form of linear combinations of different powers of the state operator.

Let us consider a mechanical system described by the following linear partial differential equation with two control forces:

$$
\begin{aligned}
\frac{\partial^2 x(z,t)}{\partial t^2} + \frac{\partial^4 x(z,t)}{\partial z^4} + \alpha \frac{\partial^5 x(z,t)}{\partial z^4 \partial t} - \beta \frac{\partial^3 x(z,t)}{\partial z^2 \partial t} \\
+ \gamma \frac{\partial^2 x(z,t)}{\partial z^2} = e^{z} u_1(t) - z u_2(t), \qquad (58)
\end{aligned}
$$

for $x \in (0, L_0),\ t \geq t_0,\ \alpha, \gamma > 0,\ \beta > 2$, with initial conditions

$$
x(z,0) = x_0(z), \quad \frac{\partial x(z,0)}{\partial t} = x_1(z), \quad z \in (0, L_0), \qquad (59)
$$

and boundary conditions

$$
x(0,t) = x(L_0,t) = \frac{\partial^2 x(0,t)}{\partial z^2} = \frac{\partial^2 x(L_0,t)}{\partial z^2} = 0, \qquad t \geq 0. \qquad (60)
$$

The function $x(z,t)$ is equal to the movement of the analysed elastic beam in the direction of the $y$-axis at the time moment $t \geq t_0$ and at the point $z \in (0, L_0)$. The first two terms in (58) are the only terms taken into account for the ideally springy elastic beam. The next two terms represent internal structural dampings, and the remaining fifth term represents the effect of the axial force on the beam. A more detailed description of these terms and the phenomenon they describe can be found in (Chen and Russel, 1982; Ito and Kunimatsu, 1991; Sakawa, 1984).

**9.1. Transformation of the partial differential equation.** In this section we shall transform the partial differential equation (58) into the form of an abstract differential equation (1) and then into an infinite sequence (8). Define the unbounded linear differential operator $A$ (Ito and Kunimatsu, 1988; Sakawa, 1983):

$$Ax(z) = \frac{\partial^4 x(z)}{\partial z^4}, \quad x \in D(A), \tag{61}$$

$$\begin{aligned} D(A) = \Big\{ & x(z) \in X^4([0, L_0], \mathbb{R}) : \\ & \frac{\mathrm{d}^4}{\mathrm{d}z^4} x(z) \in L^2([0, L_0], \mathbb{R}), \\ & x(0) = x(L_0) = \frac{\mathrm{d}^2 x}{\mathrm{d}z^2}(0) = \frac{\mathrm{d}^2 x}{\mathrm{d}z^2}(L_0) = 0 \Big\}, \end{aligned} \tag{62}$$

where $X^4([0, L_0], \mathbb{R})$ denotes the Sobolev space of all square integrable functions defined on the interval $[0, L_0]$ such that their first four derivatives are also square integrable. It can be proved (Ito and Kunimatsu, 1988; Sakawa, 1983) that the eigenvalues $\lambda_i$ and the eigenfunctions $\phi_i(z)$ of the operator $A$ are respectively given by

$$\lambda_i = \left(\frac{i\pi}{L_0}\right)^4, \quad \phi_i(z) = \sqrt{\frac{2}{L_0}} \sin\left(\frac{\pi i z}{L_0}\right), \quad i = 1, 2, 3, \dots \tag{63}$$

and the operator $A$ is linear, self-adjoint and positive definite. Particularly, one can define the following fractional power of the operator $A$:

$$A^{\frac{1}{2}} x = -\frac{\partial^2 x}{\partial z^2}, \tag{64}$$

$$\begin{aligned} D(A^{\frac{1}{2}}) = \Big\{ & x \in X^2([0, L_0], \mathbb{R}) : \\ & \frac{\mathrm{d}^2}{\mathrm{d}z^2} x(z) \in L^2([0, L_0], \mathbb{R}), \\ & x(0) = x(L_0) = 0 \Big\}, \end{aligned} \tag{65}$$

where $X^2$ denotes the second-order Sobolev space. The mechanical system considered does not contain delayed controls, so we have $M = 0$ and the input operator can be defined as follows:

$$\begin{aligned} B_0 u(t) &= \sum_{l=1}^{2} b_0^{(l)} u_l(t), \quad B_k \in L(U, X), \\ X &= L^2([0, L_0], \mathbb{R}), \end{aligned} \tag{66}$$

where $b_0^{(1)} = e^z$, $b_0^{(2)} = -z$. Applying the operators $A$ (61) and $B_0$ (66) to the partial differential equation (58), we obtain the following abstract, ordinary second-order differential equation with respect to $t$ in the Hilbert space $X$:

$$\begin{aligned} \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \left(\alpha A + \beta A^{\frac{1}{2}}\right) \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \left(A - \gamma A^{\frac{1}{2}}\right) x(t) \\ = B_0 u(t) \qquad t \geq t_0. \end{aligned} \tag{67}$$

Let us now check whether the abstract differential equation (67) fulfills the assumptions of the system (1) stated in Section 2. Based on (13) and (67), we can compute the coefficients of its characteristic equation:

$$\begin{aligned} f_{i1}^* &= \alpha \left(\frac{i\pi}{L_0}\right)^4 + \beta \left(\frac{i\pi}{L_0}\right)^2, \\ f_{i0}^* &= \left(\frac{i\pi}{L_0}\right)^4 - \gamma \left(\frac{i\pi}{L_0}\right)^2, \qquad i = 1, 2, 3, \dots \end{aligned} \tag{68}$$

The roots of the characteristic equation of the system (67) can be expressed as

$$\begin{aligned} s_{i1} &= \frac{-f_{i1}^* - \sqrt{f_{i1}^{*2} - 4 f_{i0}^*}}{2}, \\ s_{i2} &= \frac{-f_{i1}^* + \sqrt{f_{i1}^{*2} - 4 f_{i0}^*}}{2}, \quad i = 1, 2, 3, \dots \end{aligned} \tag{69}$$

From (68) we can see that for $\alpha, \gamma > 0$, and $\beta > 2$, the term $f_{i1}^{*2} - 4 f_{i0}^*$ is positive for each $i = 1, 2, 3, \dots$ . Thus from (69) we see that $s_{i1} \neq s_{i2}$ for each $i = 1, 2, 3, \dots$ Similarly, we can see that roots $s_{i1}$ and $s_{i2}$ have an upper limit for $i = 1, 2, 3, \dots$ . Thus the operator $A$ is the infinitesimal generator of a strongly continuous semigroup (Curtain and Zwart, 1995, pp. 17).

Taking into account that the operator (61) has only single eigenvalues (cf. (63)), we can find the sequence of the state and input matrices $A_i$, $B_{0i}$ in the infinite sequence of the finite dimensional systems (8) in the form

$$\begin{aligned} A_i &= \begin{bmatrix} 0 & 1 \\ -f_{i0}^* & -f_{i1}^* \end{bmatrix}, \\ B_{0i} &= \begin{bmatrix} 0 & 0 \\ \langle b_0^{(1)}, \phi_i \rangle_X & \langle b_0^{(2)}, \phi_i \rangle_X \end{bmatrix}, \\ & i = 1, 2, 3, \dots, \end{aligned} \tag{70}$$

where $\phi_i$ is given by (63). The scalar products $\langle b_0^{(l)}, \phi_i \rangle_X$ can be calculated as follows:

$$\begin{aligned} \langle b_0^{(1)}, \phi_i \rangle_X &= \sqrt{\frac{2}{L_0}} \int_0^{L_0} e^z \sin\left(\frac{\pi i z}{L_0}\right) \mathrm{d}z \\ &= \frac{\sqrt{2 L_0} \pi i}{L_0^2 + \pi^2 i^2} \left[1 - e^{L_0} (-1)^i\right], \\ & i = 1, 2, 3, \dots, \end{aligned} \tag{71}$$

$$\langle b_0^{(2)}, \phi_i \rangle_X = \sqrt{\frac{2}{L_0}} \int_0^{L_0} -z \sin\left(\frac{\pi i z}{L_0}\right) \mathrm{d}z$$
$$= \sqrt{\frac{2}{L_0}} \frac{L_0^2 (-1)^i}{\pi i}, \quad i = 1, 2, 3, \dots . \quad (72)$$

Based on (71) and (72), the matrix $B_{0i}^*$ can be obtained from (32):

$$B_{0i}^* = \left[ \begin{array}{cc} \langle b_0^{(1)}, \phi_i \rangle_X & \langle b_0^{(2)}, \phi_i \rangle_X \end{array} \right],$$
$$i = 1, 2, 3, \dots . \quad (73)$$

We have already transformed the analysed mechanical system (58) to the form (8) required by all the controllability theorems proved in this article. So let us analyse whether or not that system is approximately controllable with two kinds of admissible control sets.

**9.2. Unconstrained approximate controllability.** We shall test the unconstrained approximate controllability of the mechanical system (58) by Theorem 2. In this case the infinite sequence of the equations (34) has the following form:

$$\operatorname{rank} \left[ \begin{array}{cc} \frac{\sqrt{2L_0}\pi i}{L_0^2 + \pi^2 i^2} \left[1 - e^{L_0} (-1)^i\right] & \sqrt{\frac{2}{L_0}} \frac{L_0^2 (-1)^i}{\pi i} \end{array} \right]$$
$$= 1, \quad i = 1, 2, 3, \dots . \quad (74)$$

We see that the infinite sequence of the equations (74) holds true for each $i = 1, 2, 3, \dots$ , so the mechanical system (58) is approximately controllable.

**9.3. Approximate controllability with nonnegative cone-type constraints.** We shall verify this kind of controllability of the mechanical system (58) by Theorem 4. Condition (i) is fulfilled for each $i = 1, 2, 3, \dots$ if we choose $w_i = 0$. Condition (ii) requires the convex hull of the admissible control set $U$ not to have a nonempty interior. Condition (iii) of Theorem 4 is equivalent to Theorem 2 and was verified in the previous section. Moreover, as has already been pointed out,

$$f_{i1}^{*2} - 4 f_{i0}^* > 0, \quad i = 1, 2, 3, \dots . \quad (75)$$

Accordingly, all roots $s_{i1}, s_{i2}$ are real. Thus Condition (iv) of Theorem 4 is equivalent to the requirement that for each $i = 1, 2, 3, \dots$ in the input matrix $B_{0i}$ (70) there must exist a pair of scalar products of opposite signs. Considering the form of the matrix $B_{0i}$ (70) and the form of the scalar products (71) and (72), Condition (iv) of Theorem 4 is equivalent to

$$\left[1 - e^{L_0} (-1)^i\right] (-1)^i < 0, \quad \forall i = 1, 2, 3, \dots . \quad (76)$$

This inequality is true for both odd and even $i$, and hence Condition (iv) is fulfilled.

Now, let us verify Condition (v). From (68) we can see that

$$f_{i1}^* > 0, \quad i = 1, 2, 3, \dots . \quad (77)$$

Thus from (69) and (75) we can deduce that $\operatorname{Re}[s_{i1}] < 0$ for each $i = 1, 2, 3, \dots$ Now let us verify the remaining inequality $\operatorname{Re}[s_{i2}] < 0, \ i = 1, 2, 3, \dots$ On the basis of (75), (77) and (69) we see that $\operatorname{Re}[s_{i2}] < 0, \ i = 1, 2, 3, \dots$ if and only if

$$f_{i0}^* > 0, \quad i = 1, 2, 3, \dots . \quad (78)$$

From (68) we can deduce that the infinite sequence of inequalities (78) is fulfilled if and only if $\gamma < \pi^2 / L_0^2$ .

**9.4. Summary of the example.** The elastic beam (58) with the conditions (59) and (60) is

- approximately controllable,
- approximately controllable with nonnegative cone-type constraints if and only if the convex hull of the admissible cone control set $U$ has nonempty interior and the parameter $\gamma$ satisfies $\gamma < \pi^2 / L_0^2$.

## 10. Conclusions

We obtained general conditions for various types of controllability for infinite dimensional systems, at once for any order of the system. This problem turned out to be more sophisticated than for fixed equation order. This was possible by analyzing the $n$-th order linear system in the Frobenius form, generating a Jordan transition matrix in the Vandermonde form, and making use of Chen's theorem. To accomplish the task, we introduced a general analytical form of the inverse Vandermonde matrix, known from linear algebra, into controllability theory. The obtained theorems of approximate controllability without constraints, with cone-type constraints, and with delays in control hold true for any order of the verified infinite dimensional dynamic system. This is a new result in controllability theory.

Moreover, it should be pointed out that the presented methods can be easily adapted to the analysis of other dynamic properties of the $n$-th order system considered, i.e., observability, attainability, stability and optimal control.

Fulfilling the condition (15) does not depend on the class of the equation considered, but on the values of the equation coefficients. A possible direction of further investigations can be a generalization of the presented results to the case of arbitrary eigenvalue multiplicities of the state operator.

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