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POSITIVE 2D DISCRETE–TIME LINEAR LYAPUNOV SYSTEMS

PRZEMYSŁAW PRZYBOROWSKI, TADEUSZ KACZOREK

Institute of Control and Industrial Electronics
Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland
e-mail: {przyborp,kaczorek}@isep.pw.edu.pl

Two models of positive 2D discrete-time linear Lyapunov systems are introduced. For both the models necessary and sufficient conditions for positivity, asymptotic stability, reachability and observability are established. The discussion is illustrated with numerical examples.

Keywords: positivity, Lyapunov systems, reachability, observability.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser (1975), Fornasini-Marchesini (1976; 1978) and Kurek (1985). The models were extended for positive systems in (Kaczorek, 1996; 2001; 2005; Valcher, 1997). An overview of 2D linear systems theory is given in (Bose, 1982; Bose *et al.*, 2003; Gałkowski, 2001; Kaczorek, 1985), and some recent results in positive systems were given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2001).

Reachability and minimum energy control of positive 2D systems with one delay in states were considered in (Kaczorek, 2005). Controllability of positive dynamical systems was investigated by Klamka (1991; 2002; 2005). Controllability and minimum energy control of linear 2D systems were considered in (Klamka, 1996a; 1996b; 1997a; 1997b; 1997d; 1999b) and of nonlinear 2D systems in (Klamka 1997c; 1999a; 1999c). Controllability with constrained controls of linear and nonlinear 2D systems was investigated in (Klamka, 1998a; 1998b; 1998c).

The notion of an internally positive 2D system (model) with delays in states and in inputs (systems of order higher than one) was introduced, and necessary and sufficient conditions for internal positivity, reachability, controllability, observability and the minimum energy control problem were established in (Kaczorek, 2006b).

The realization problem for 1D positive discrete-time systems with delays was analyzed in (Kaczorek, 2003; 2006a) and for 2D positive systems in (Kaczorek, 2004). Stability of positive linear discrete-time systems with delays was considered in (Busłowicz, 2006).

Internal stability and asymptotic behavior of 2D positive systems were investigated by Valcher (1997), and asymptotic stability of positive 2D linear systems was investigated in (Kaczorek, 2008a; 2008b). An LMI approach to checking stability of positive 2D systems was proposed by Twardy (2007), with generalizations to positive 2D systems by delays in (Kaczorek, 2008c).

Controllability and observability of Lyapunov systems were investigated by Murty Apparao (2005). Positive discrete-time and continuous-time Lyapunov systems were considered in (Kaczorek, 2007; Kaczorek and Przyborowski, 2007a; 2007e; 2008). Positive linear time-varying Lyapunov systems were investigated in (Kaczorek and Przyborowski, 2007b). Discrete-time and continuous-time Lyapunov cone systems were considered in (Kaczorek and Przyborowski, 2007c; Przyborowski and Kaczorek, 2008). Positive discrete-time Lyapunov systems with delays were investigated in (Kaczorek and Przyborowski, 2007d).

Positive fractional discrete-time Lyapunov systems were investigated in (Przyborowski, 2008a; Przyborowski and Kaczorek, 2008) and fractional discrete-time cone-systems in (Przyborowski, 2008b; Przyborowski and Kaczorek, 2008).

In this paper, the notion of positive 2D discrete-time linear Lyapunov systems described by two different models will be introduced. For both the models necessary and sufficient conditions for positivity, asymptotic stability, reachability and observability will be established. The discussion will be illustrated with numerical examples. To the best of the authors' knowledge, those problems have not been considered yet.

2. Preliminaries

Let $\mathbb{R}^{n\times m}$ be the set of real $n \times m$ matrices, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, and let $\mathbb{R}_+^{n\times m}$ be the set of real $n \times m$ matrices with nonnegative entries. The set of nonnegative integers will be denoted by $\mathbb{Z}_+$.

Definition 1. The Kronecker product $A \otimes B$ of matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is the block matrix (Kaczorek, 1998)

$$A \otimes B = [a_{ij}B]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathbb{R}^{mp\times nq}. \tag{1}$$

Lemma 1. (Kaczorek, 1998) *Consider the equation*

$$AXB = C, \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times p}$, $C \in \mathbb{R}^{m\times p}$, $X \in \mathbb{R}^{n\times q}$. It is equivalent to the following one:

$$(A \otimes B^T)x = c, \tag{3}$$

where

$$x := [x_1, x_2, \dots, x_n]^T, \quad c := [c_1, c_2, \dots, c_m]^T,$$

and x_i and c_i are the i-th rows of the matrices X and C, respectively.

Lemma 2. (Kaczorek, 1998) *If $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ and $\mu_1, \mu_2, \dots \mu_n$ are the eigenvalues of the matrix $B \in \mathbb{R}^{n\times n}$, then $\lambda_i + \mu_j$ for $i, j = 1, 2, \dots, n$ are the eigenvalues of the matrix*

$$\bar{A} = A \otimes I_n + I_n \otimes B^T.$$

3. 2D Lyapunov system

Definition 2. The system described by the equations

$$\begin{aligned}\begin{bmatrix} X^h_{i+1,j} \\ X^v_{i,j+1} \end{bmatrix} &= \begin{bmatrix} A^0_{11} & A^0_{12} \\ A^0_{21} & A^0_{22} \end{bmatrix} \begin{bmatrix} X^h_{i,j} \\ X^v_{i,j} \end{bmatrix} \\ &+ \begin{bmatrix} X^h_{i,j} \\ X^v_{i,j} \end{bmatrix} \begin{bmatrix} A^1_{11} & A^1_{12} \\ A^1_{21} & A^1_{22} \end{bmatrix} \\ &+ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U_{ij},\end{aligned} \tag{4a}$$

$$Y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} X^h_{i,j} \\ X^v_{i,j} \end{bmatrix} + DU_{ij}, \quad i, j \in \mathbb{Z}_+ \tag{4b}$$

is called a *2D discrete-time linear Lyapunov system,* where $X^h_{i,j} \in \mathbb{R}^{n_1\times n}$ and $X^v_{i,j} \in \mathbb{R}^{n_2\times n}$ are respectively the horizontal and vertical state-space matrices at the point (i, j), $U_{ij} \in \mathbb{R}^{m\times n}$ and $Y_{ij} \in \mathbb{R}^{p\times n}$ are respectively the input and the output matrices, $A^r_{kl} \in \mathbb{R}^{n_k\times n_l}$ for $k, l = 1, 2$ and $r = 0, 1$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$, $D \in \mathbb{R}^{p\times m}$, $n = n_1 + n_2$.

The boundary conditions for (4a) have the form

$$X^h_{0j},\ j \in \mathbb{Z}_+ \ \text{and}\ X^v_{i0},\ i \in \mathbb{Z}_+. \tag{5}$$

Lemma 3. *The Lyapunov system (4) can be transformed to the equivalent standard 2D discrete-time, nm-input and pn-output, linear system described by the Roesser model in the form (Kaczorek, 2001)*

$$\begin{aligned}\begin{bmatrix} \bar{x}^h_{i+1,j} \\ \bar{x}^v_{i,j+1} \end{bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} \\ &+ \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \bar{u}_{ij},\end{aligned} \tag{6a}$$

$$\bar{y}_{ij} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^h_{i,j} \\ \bar{x}^v_{i,j} \end{bmatrix} + \bar{D}\bar{u}_{ij}, \quad i, j \in \mathbb{Z}_+, \tag{6b}$$

where $\bar{x}^h_{i,j} \in \mathbb{R}^{(n_1\cdot n)}$ and $\bar{x}^v_{i,j} \in \mathbb{R}^{(n_2\cdot n)}$ are respectively the horizontal and vertical state-space vectors at the point (i, j), $\bar{u}_{ij} \in \mathbb{R}^{(m\cdot n)}$ and $\bar{y}_{ij} \in \mathbb{R}^{(p\cdot n)}$ are respectively the input and output vectors, $\bar{A}_{kl} \in \mathbb{R}^{(n_k\cdot n)\times(n_l\cdot n)}$, for $k, l = 1, 2$, $\bar{B}_1 \in \mathbb{R}^{(n\cdot n_1)\times(n\cdot m)}$, $\bar{B}_2 \in \mathbb{R}^{(n\cdot n_2)\times(n\cdot m)}$, $\bar{C}_1 \in \mathbb{R}^{(p\cdot n)\times(n\cdot n_1)}$, $\bar{C}_2 \in \mathbb{R}^{(p\cdot n)\times(n\cdot n_2)}$, $\bar{D} \in \mathbb{R}^{(p\cdot n)\times(m\cdot n)}$.

Proof. The transformation is based on Lemma 1. The matrices

$$X_{i,j} = \begin{bmatrix} X^h_{i,j} \\ X^v_{i,j} \end{bmatrix}, \quad U_{i,j}, \quad Y_{i,j}$$

are transformed into the vectors

$$\bar{x}_{i,j} = \begin{bmatrix} X_{i,j}^1 & X_{i,j}^2 & \dots & X_{i,j}^n \end{bmatrix}^T,$$
$$\bar{u}_{i,j} = \begin{bmatrix} U_{i,j}^1 & U_{i,j}^2 & \dots & U_{i,j}^m \end{bmatrix}^T,$$
$$\bar{y}_{i,j} = \begin{bmatrix} Y_{i,j}^1 & Y_{i,j}^2 & \dots & Y_{i,j}^p \end{bmatrix}^T,$$

where $X_{i,j}^k, U_{i,j}^k, Y_{i,j}^k$ denote the k-th rows of the matrices $X_{i,j}, U_{i,j}, Y_{i,j}$, respectively.

The matrices of (6) are

$$\begin{aligned} \bar{A}_{11} &= A_{11}^0 \otimes I_n + I_{n_1} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T, \\ \bar{A}_{12} &= A_{12}^0 \otimes I_n, \\ \bar{A}_{22} &= A_{22}^0 \otimes I_n + I_{n_2} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T, \\ \bar{A}_{21} &= A_{21}^0 \otimes I_n, \\ \bar{B}_1 &= B_1 \otimes I_n, \quad \bar{B}_2 = B_2 \otimes I_n, \\ \bar{C}_1 &= C_1 \otimes I_n, \quad \bar{C}_2 = C_2 \otimes I_n, \\ \bar{D} &= D \otimes I_n. \end{aligned} \tag{7}$$

■

Definition 3. The *transition matrix* $\bar{T}_{i,j}$ is defined by (Kaczorek, 2001)

$$\bar{T}_{i,j} = \begin{cases} I_n & \text{for} \quad i,j=0, \\ \bar{T}_{1,0}\bar{T}_{i-1,j} + \bar{T}_{0,1}\bar{T}_{i,j-1} & \text{for} \quad i,j \in Z_+, \\ 0 \quad \text{(zero matrix)} & \text{for} \quad i<0 \\ & \text{and/or} \quad j<0, \end{cases} \tag{8}$$

where

$$\bar{T}_{1,0} = \begin{bmatrix} A_{11}^0 \otimes I_n + I_{n_1} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T & A_{12}^0 \otimes I_n \\ 0 & 0 \end{bmatrix},$$

$$\bar{T}_{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21}^0 \otimes I_n & A_{22}^0 \otimes I_n + I_{n_2} \otimes \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}^T \end{bmatrix}.$$

4. Positive 2D Lyapunov systems and their asymptotic stability

4.1. Positive 2D Lyapunov systems

Definition 4. The system (4) is called *(internally) positive* if $X_{i,j}^h \in \mathbb{R}_+^{n_1\times n}$, $X_{i,j}^v \in \mathbb{R}_+^{n_2\times n}$ and $Y_{ij} \in \mathbb{R}_+^{p\times n}$ for any nonnegative boundary conditions X_{0j}^h, X_{i0}^v and all input sequences $U_{ij} \in \mathbb{R}_+^{m\times n}$, $i,j \in \mathbb{Z}_+$.

Definition 5. A matrix

$$M = [m_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

is called a *Metzler matrix* if $m_{ij} \in \mathbb{R}$ for $i=j$ and $m_{ij} \geq 0$ for $i \neq j$.

Theorem 1. *The system (4) is positive if and only if*

$$A_{11}^0 = \left[a_{ij}^{011}\right]_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_1}}, \quad A_{22}^0 = \left[a_{ij}^{022}\right]_{\substack{i=1,\dots,n_2 \\ j=1,\dots,n_2}},$$
$$A_{11}^1 = \left[a_{ij}^{111}\right]_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_1}}, \quad A_{22}^1 = \left[a_{ij}^{122}\right]_{\substack{i=1,\dots,n_2 \\ j=1,\dots,n_2}} \tag{9a}$$

are Metzler matrices satisfying

$$\begin{aligned} &a_{kk}^{011} + a_{ll}^{111} \geq 0 \ \textit{for} \ k,l=1,\dots,n_1, \\ &a_{kk}^{022} + a_{ll}^{111} \geq 0 \ \textit{for} \ k=1,\dots,n_2\,;\ l=1,\dots,n_1, \\ &a_{kk}^{011} + a_{ll}^{122} \geq 0 \ \textit{for} \ k=1,\dots,n_1\,;\ l=1,\dots,n_2, \\ &a_{kk}^{022} + a_{ll}^{122} \geq 0 \ \textit{for} \ k,l=1,\dots,n_2, \end{aligned} \tag{9b}$$

and

$$\begin{aligned} &A_{kl}^r \in \mathbb{R}_+^{n_k\times n_l} \ \textit{for}\ k,l=1,2,\ k\neq l;\ r=0,1, \\ &B_1 \in \mathbb{R}_+^{n_1\times m}, \quad B_2 \in \mathbb{R}_+^{n_2\times m}, \\ &C_1 \in \mathbb{R}_+^{p\times n_1}, \quad C_2 \in \mathbb{R}_+^{p\times n_2}, \\ &D \in \mathbb{R}_+^{p\times m}. \end{aligned} \tag{9c}$$

Proof. The 2D Lyapunov system (4) is positive if, and only if, the equivalent 2D standard system (6) is positive. By the theorem of the positivity of the 2D standard discrete-time system described by the Roesser model (Kaczorek, 2001),

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}, \quad \bar{D}$$

have to be matrices with nonnegative entries. From (7) the hypothesis of Theorem 1 follows. ■

4.2. Asymptotic stability of 2D positive Lyapunov systems. Consider the positive 2D autonomous Lyapunov system described by

$$\begin{bmatrix} X_{i+1,j}^h \\ X_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} \begin{bmatrix} X_{i,j}^h \\ X_{i,j}^v \end{bmatrix} + \begin{bmatrix} X_{i,j}^h \\ X_{i,j}^v \end{bmatrix} \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{10}$$

where $X_{i,j}^h \in \mathbb{R}_+^{n_1\times n}$,$X_{i,j}^v \in \mathbb{R}_+^{n_2\times n}$ and the matrices $A_{kl}^r \in \mathbb{R}^{n_k\times n_l}$ for $k,l=1,2$ and $r=0,1$, satisfying the conditions (9).

Definition 6. The positive 2D Lyapunov system (10) is called *asymptotically stable* if for any bounded boundary conditions $X_{i,0} \in \mathbb{R}_+^{n\times n}$, $i \in \mathbb{Z}_+$, $X_{0,j} \in \mathbb{R}_+^{n\times n}$, $j \in \mathbb{Z}_+$ we have

$$\lim_{i,j\to\infty} X_{i,j} = 0. \tag{11}$$

Theorem 2. *Assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix*

$$\begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix}$$

and $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of the matrix

$$\begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}.$$

The system (10) is stable if and only if

$$|\lambda_i + \beta_j| < 1 \quad \text{for} \;\; i, j = 1, 2, \dots, n. \tag{12}$$

Proof. Any 2D Lyapunov system is asymptotically stable if, and only if, the equivalent 2D standard system is asymptotically stable. From (Kaczorek, 2008a), we have that the eigenvalues of the matrix

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$$

must have moduli less than one. Therefore, from Lemma 3 and (7) the hypothesis of Theorem 2 follows. ■

5. Reachability and observability of 2D positive systems

5.1. Reachability

Definition 7. The positive 2D Lyapunov system (4) is called *reachable* at a point $(h, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if for every $X_f \in \mathbb{R}_+^{n\times n}$ there exists an input sequence $U_{ij} \in \mathbb{R}_+^{m\times n}$ for

$$(i, j) \in H_{hk} := \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \\ 0 \le i \le h,\ 0 \le j \le k,\ i + j \ne h + k\}$$

that steers the state of the system from the zero boundary conditions (5) to the final state X_f, i.e., $X_{hk} \in X_f$.

Theorem 3. *The positive 2D Lyapunov system (4) is reachable at a point (h, k) if and only if*
(a) For

$$A_1 = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}$$

satisfying the condition $XA_1 = A_1X$, i.e., $A_{11}^1 = aI_{n_1}$, $A_{22}^1 = aI_{n_2}$, $a \in \mathbb{R}$, $A_{12}^1 = 0$ and $A_{21}^1 = 0$, the matrix

$$R_{hk} = [M_{h,k}\ M_{h-1,k}\ M_{h,k-1}\ \cdots\ M_{1,0}\ M_{0,1}] \tag{13}$$

contains n linearly independent monomial columns (the matrix built from these columns has only one positive element in each row and in each column and the remaining elements are zero), where

$$M_{i,j} = T_{i-1,j} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i,j-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{14}$$

and $T_{i,j}$ is the transition matrix defined in (8) with

$$\mathrm{T}_{1,0} = \begin{bmatrix} A_{11}^0 + A_{11}^1 & A_{12}^0 \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{T}_{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21}^0 & A_{22}^0 + A_{22}^1 \end{bmatrix}. \tag{15}$$

(b) For $A_1 \ne aI_n$ and $a \in \mathbb{R}$, if and only if the matrix

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

contains n linearly independent monomial columns.

Proof. From Lemma 3 and (Kaczorek, 2001) it follows that the positive 2D Lyapunov system (4) is reachable at the point (h, k) if and only if the matrix

$$\bar{R}_{hk} = \left[\bar{M}_{h,k}\ \bar{M}_{h-1,k}\ \bar{M}_{h,k-1}\ \cdots\ \bar{M}_{1,0}\ \bar{M}_{0,1}\right] \tag{16}$$

contains n^2 linearly independent monomial columns, where

$$\bar{M}_{i,j} = \bar{T}_{i-1,j} \begin{bmatrix} B_1 \otimes I_n \\ 0 \end{bmatrix} + \bar{T}_{i,j-1} \begin{bmatrix} 0 \\ B_2 \otimes I_n \end{bmatrix} \tag{17}$$

and $\bar{T}_{i,j}$ is the transition matrix defined in (8).

In Case (a), taking into account the assumptions, from (16), (17), (8) we obtain

$$\bar{T}_{i,j} = T_{i,j} \otimes I_n, \\ \bar{M}_{i,j} = M_{i,j} \otimes I_n, \\ \bar{R}_{h,k} = R_{h,k} \otimes I_n.$$

Therefore, in this case, (16) contains n^2 linearly independent monomial columns if and only if (13) contains n linearly independent monomial columns.

In Case (b), from (17) we have

$$\bar{M}_{1,0} = \begin{bmatrix} B_1 \otimes I_n \\ 0 \end{bmatrix}, \quad \bar{M}_{0,1} = \begin{bmatrix} 0 \\ B_2 \otimes I_n \end{bmatrix}$$

so if the matrix $\left[\begin{smallmatrix} B_1 & 0 \\ 0 & B_2 \end{smallmatrix}\right]$ contains n linearly independent monomial columns, then $\bar{R}_{h,k}$ contains n^2 linearly independent monomial columns and the system is reachable. If the matrix $\left[\begin{smallmatrix} B_1 & 0 \\ 0 & B_2 \end{smallmatrix}\right]$ contains $r < n$ linearly independent monomial columns, then from (17) it follows that each of the matrices $\bar{M}_{1,1}, \cdots, \bar{M}_{h,k}$ contains no more than rn linearly independent monomial columns which are linearly dependent with monomial columns of the matrix $\left[\bar{M}_{1,0} \quad \bar{M}_{0,1}\right]$, because the matrices $\bar{T}_{i,j}$ and $B \otimes I_n$ have nonnegative entries. Therefore, the system is not reachable. ■

5.2. Observability

Definition 8. The positive 2D Lyapunov system (4) is called *observable* at a point $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if $X_{00} \in \mathbb{R}_+^{n\times n}$ can be uniquely determined from the knowledge of the output $Y_{i,j}$, caused by the nonzero boundary conditions in the form $X_{00} \neq 0$ and $X_{0j}^h = 0,\ 1 \le j \le k$, $X_{i0}^v = 0,\ 1 < i \le h$ and $U_{i,j} = 0,\ (i,j) \in H_{hk}$.

Theorem 4. *The positive 2D Lyapunov system (4) is observable at the point (h,k) if and only if*
(a) For

$$A_1 = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}$$

satisfying the condition $XA_1 = A_1X$, i.e., $A_{11}^1 = aI_{n_1}$,$A_{22}^1 = aI_{n_2}$,$a \in \mathbb{R}$ and $A_{12}^1 = 0$, $A_{21}^1 = 0$, the matrix

$$O_{hk} = \begin{bmatrix} C \\ CT_{10} \\ CT_{01} \\ \vdots \\ CT_{i,j} \\ \vdots \\ CT_{h,k} \end{bmatrix} \tag{18}$$

contains n linearly independent monomial rows, where $C = [C_1 \quad C_2]$ and $T_{i,j}$ is the transition matrix defined in (8) with

$$\mathrm{T}_{1,0} = \begin{bmatrix} A_{11}^0 + A_{11}^1 & A_{12}^0 \\ 0 & 0 \end{bmatrix},$$
$$\mathrm{T}_{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21}^0 & A_{22}^0 + A_{22}^1 \end{bmatrix}. \tag{19}$$

(b) For $A_1 \neq aI_n$ and $a \in \mathbb{R}$, if and only if the matrix C contains n linearly independent monomial rows.

Proof. From Lemma 3 and (Kaczorek, 2001) it follows that the positive 2D Lyapunov system (4) is observable at a point (h,k) if and only if the matrix

$$\bar{O}_{hk} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{T}_{10} \\ \bar{C}\bar{T}_{01} \\ \vdots \\ \bar{C}\bar{T}_{i,j} \\ \vdots \\ \bar{C}\bar{T}_{h,k} \end{bmatrix} \tag{20}$$

contains n^2 linearly independent monomial columns, where $\bar{T}_{i,j}$ is the transition matrix defined in (8).

In Case (a), taking into account the assumptions, from (20), (8) and the fact that $\bar{C} = C \otimes I_n$, we obtain

$$\bar{T}_{i,j} = T_{i,j} \otimes I_n, \quad \bar{O}_{h,k} = O_{h,k} \otimes I_n.$$

Therefore, in this case, (20) contains n^2 linearly independent monomial columns if and only if (18) contains n linearly independent monomial columns.

In Case (b), if the matrix C contains n linearly independent monomial columns, then $\bar{O}_{h,k}$ contains n^2 linearly independent monomial columns and the system is observable. If the matrix C contains $r < n$ linearly independent monomial columns, then it follows that each of the matrices $\bar{C}\bar{T}_{10}, \ldots, \bar{C}\bar{T}_{h,k}$ contains no more than rn linearly independent monomial columns which are linearly dependent with monomial columns of the matrix C because the matrices $\bar{T}_{i,j}$ and $\bar{C}$ are the matrices with nonnegative entries. Therefore the system is not observable. ■

6. 2D general Lyapunov system

Definition 9. The system described by the equations

$$\begin{aligned} X_{i+1,j+1} &= A_0^0 X_{i,j} + X_{i,j}A_0^1 + A_1^0 X_{i+1,j} \\ &\quad + X_{i+1,j}A_1^1 + A_2^0 X_{i,j+1} + X_{i,j+1}A_2^1 \\ &\quad + B_0 U_{i,j} + B_1 U_{i+1,j} + B_2 U_{i,j+1}, \end{aligned} \tag{21a}$$
$$Y_{ij} = CX_{i,j} + DU_{ij}, \qquad i,j \in \mathbb{Z}_+ \tag{21b}$$

is called a *general 2D discrete-time linear Lyapunov system*, where $X_{i,j} \in \mathbb{R}^{n\times n}$ is the state-space matrix at the point (i,j), $U_{ij} \in \mathbb{R}^{m\times n}$ and $Y_{ij} \in \mathbb{R}^{p\times n}$ are respectively the input and the output matrices, $A_k^l \in \mathbb{R}^{n\times n}$ for $k = 0,1,2$, $l = 0,1$, $B_r \in \mathbb{R}^{n\times m}$ for $r = 0,1,2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

The boundary conditions for (21a) have the form

$$X_{0j},\ j \in \mathbb{Z}_+ \quad \text{and} \quad X_{i0},\ i \in \mathbb{Z}_+. \tag{22}$$

Lemma 4. *The Lyapunov system (21) can be transformed to the equivalent standard 2D discrete-time, nm-input and pn-output, linear system described by the general model in the form (Kaczorek, 2001)*

$$\begin{aligned}\bar{x}_{i+1,j+1} &= \bar{A}_0\bar{x}_{i,j} + \bar{A}_1\bar{x}_{i+1,j} + \bar{A}_2\bar{x}_{i,j+1} \\ &\quad +\bar{B}_0\bar{u}_{i,j} + \bar{B}_1\bar{u}_{i+1,j} \qquad (23a)\\ &\quad +\bar{B}_2\bar{u}_{i,j+1}, \qquad (23b)\\ \bar{y}_{ij} &= \bar{C}\bar{x}_{i,j} + \bar{D}\bar{u}_{ij} \qquad i,j\in\mathbb{Z}_+, \qquad (23c)\end{aligned}$$

where $\bar{x}_{i,j} \in \mathbb{R}^{n^2\times n^2}$ is the state-space vector at the point (i,j), $\bar{u}_{ij} \in \mathbb{R}^{(m\cdot n)}$ and $\bar{y}_{ij} \in \mathbb{R}^{(p\cdot n)}$ are respectively the input and the output vectors, $A_k \in \mathbb{R}^{n^2\times n^2}$ for $k = 0,1,2$, $B_r \in \mathbb{R}^{n^2\times(m\cdot n)}$ for $r = 0,1,2$, $C \in \mathbb{R}^{(p\cdot n)\times n^2}$, $D \in \mathbb{R}^{(p\cdot n)\times(m\cdot n)}$.

The proof is similar to that of Lemma 3. The matrices of (23) are

$$\begin{aligned}\bar{A}_0 &= A_0^0 \otimes I_n + I_n \otimes A_0^{1T},\\ \bar{A}_1 &= A_1^0 \otimes I_n + I_n \otimes A_1^{1T}\\ \bar{A}_2 &= A_2^0 \otimes I_n + I_n \otimes A_2^{1T},\\ \bar{B}_0 &= B_0 \otimes I_n, \quad \bar{B}_1 = B_1 \otimes I_n,\\ \bar{B}_2 &= B_2 \otimes I_n, \quad \bar{C} = C \otimes I_n, \quad \bar{D} = D \otimes I_n.\end{aligned} \qquad (24)$$

Definition 10. The *transition matrix* $\bar{T}_{i,j}$ for (23) is defined by (Kaczorek, 2001)

$$\bar{T}_{i,j} = \begin{cases} I_n & \text{for } i,j=0,\\ \bar{A}_0\bar{T}_{i-1,j-1} & \\ \quad +\bar{A}_1\bar{T}_{i,j-1} + \bar{A}_2\bar{T}_{i-1,j} & \text{for } i,j\in Z_+\\ 0 \text{ (zero matrix)} & \text{for } i<0 \\ & \text{and/or } j<0.\end{cases} \qquad (25)$$

7. Positive general 2D Lyapunov systems and their asymptotic stability

7.1. Positive general 2D Lyapunov systems

Definition 11. The system (21) is called *(internally) positive* if $X_{i,j} \in \mathbb{R}_+^{n\times n}$ and $Y_{ij} \in \mathbb{R}_+^{p\times n}$ for any nonnegative boundary conditions $X_{0j} \in \mathbb{R}_+^{n\times n}$, $X_{i0} \in \mathbb{R}_+^{n\times n}$ and all input sequences $U_{ij} \in \mathbb{R}_+^{m\times n}$, $i,j \in \mathbb{Z}_+$.

Theorem 5. *The system (21) is positive if and only if*

$$A_k^l = \left[a_{ij}^{kl}\right]_{\substack{i=1,\dots,n\\ j=1,\dots,n}}, \quad k=0,1,2, \quad l=0,1 \qquad (26a)$$

are Metzler matrices satisfying the conditions

$$xa_{pp}^{k0} + a_{rr}^{k1} \geq 0 \text{ for } p,r=1,\dots,n \text{ and } k=0,1,2, \qquad (26b)$$

where

$$B_0 \in \mathbb{R}_+^{n\times m}, \quad B_1 \in \mathbb{R}_+^{n\times m}, \quad B_2 \in \mathbb{R}_+^{n\times m},$$
$$C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \qquad (26c)$$

Proof. The 2D Lyapunov system (21) is positive if, and only if, the equivalent 2D standard system (23) is positive. By the theorem of the positivity of the 2D standard discrete-time system described by the general model (Kaczorek, 2001), $\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{B}_0, \bar{B}_1, \bar{B}_2$ $\bar{C}$ and $\bar{D}$ have to be matrices with nonnegative entries. The hypothesis of Theorem 5 follows from (24). ■

7.2. Asymptotic stability of general 2D positive Lyapunov systems. Consider the positive 2D autonomous Lyapunov system described by

$$\begin{aligned}X_{i+1,j+1} &= A_0^0X_{i,j} + X_{i,j}A_0^1\\ &\quad + A_1^0X_{i+1,j} + X_{i+1,j}A_1^1\\ &\quad + A_2^0X_{i,j+1} + X_{i,j+1}A_2^1, \quad i,j\in\mathbb{Z}_+,\end{aligned} \qquad (27)$$

where $X_{i,j} \in \mathbb{R}_+^{n\times n}$, with the matrices $A_k^l \in \mathbb{R}^{n\times n}$ for $k=0,1,2$ and $l=0,1$ satisfying the conditions (26).

Definition 12. The positive 2D Lyapunov system (27) is called *asymptotically stable* if for any bounded boundary conditions $X_{i,0} \in \mathbb{R}_+^{n\times n}$, $i\in\mathbb{Z}_+$, $X_{0,j} \in \mathbb{R}_+^{n\times n}$, $j\in\mathbb{Z}_+$,

$$\lim_{i,j\to\infty} X_{i,j} = 0. \qquad (28)$$

Theorem 6. *Assume that $\lambda_1, \lambda_2, \dots, \lambda_{n^2}$ are the eigenvalues of the matrix*

$$\begin{bmatrix} A_1^0 + A_2^0 & A_0^0\\ I_n & 0\end{bmatrix}$$

and $\mu_1, \mu_2, \dots, \mu_{n^2}$ are the eigenvalues of the matrix

$$\begin{bmatrix} A_1^1 + A_2^1 & A_0^1\\ I_n & 0\end{bmatrix}.$$

The system (27) is stable if and only if

$$|\lambda_i + \beta_j| < 1 \quad \text{for} \quad i,j = 1,2,\dots,n^2. \qquad (29)$$

Proof. The 2D Lyapunov system is asymptotically stable if, and only if, the equivalent 2D standard system is asymptotically stable. From (Kaczorek, 2008a) we have that the eigenvalues of the matrix

$$\begin{bmatrix} \bar{A}_1 + \bar{A}_2 & \bar{A}_0\\ I_{n^2} & 0\end{bmatrix}$$

must have moduli less than one. Therefore, from Lemma 4 and (24), the hypothesis of Theorem 6 follows. ■

8. Reachability and observability of 2D positive systems

8.1. Reachability

Definition 13. The positive 2D Lyapunov system (21) is called reachable at a point $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if for every $X_f \in \mathbb{R}_+^{n\times n}$ there exists an input sequence $U_{ij} \in \mathbb{R}_+^{m\times n}$, $(i,j) \in H_{hk}$ that steers the state of the system from the zero boundary conditions (22) to the final state X_f, i.e., $X_{hk} \in X_f$.

Theorem 7. *The positive 2D Lyapunov system (21) is reachable at a point (h,k), $h,k > 2$ if, and only if,*

(a) For A_l^1 satisfying the condition $XA_l^1 = A_l^1 X$, i.e. $A_l^1 = a_l I_n$, $a_l \in \mathbb{R}$, $l = 0,1,2$, if and only if the matrix

$$R_{hk} = \big[M_0,\, M_1^1, \dots, M_h^1, M_1^2, \dots, M_k^2, M_{11}, \dots, M_{1k},\, M_{21}, \dots, M_{hk}\big] \quad (30)$$

contains n linearly independent monomial columns, where

$$\begin{aligned} M_0 &= T_{h-1,k-1}B_0,\\ M_i^1 &= T_{h-i,k-1}B_1 + T_{h-i-1,k-1}B_0, \quad i = 1,\dots,h\\ M_j^2 &= T_{h-1,k-j}B_2 + T_{h-i,k-j-1}B_0, \quad j = 1,\dots,k\\ M_{i,j} &= T_{h-i-1,k-1-1}B_0 + T_{h-i,k-j-1}B_1\\ &\quad + T_{h-i-1,k-j}B_2, \quad i = 1,\dots,h,\ j = 1,\dots,k \end{aligned} \quad (31)$$

and $T_{i,j}$ is the transition matrix defined by

$$T_{i,j} = \begin{cases} I_n & \text{for } i,j = 0,\\ \widehat{A}_0\, T_{i-1,j-1} & \\ \quad + \widehat{A}_1\, T_{i,j-1} + \widehat{A}_2\, T_{i-1,j} & \text{for } i,j \in \mathbb{Z}_+,\\ 0 \ \ (\text{zero matrix}) & \text{for } i < 0\\ & \text{and/or } j < 0, \end{cases} \quad (32)$$

$$\widehat{A}_v = A_v^0 + A_v^1, \quad v = 0,1,2.$$

(b) For $A_l \neq a_l I_n$ and $a_l \in \mathbb{R}$; $l = 0,1,2$, if and only if the matrix $[\ B_1 \quad B_2\ ]$ for $B_1 \neq 0, B_2 \neq 0$ (B_0 for $B_1 = B_2 = 0$) contains n linearly independent monomial columns.

Proof. From Lemma 4 and (Kaczorek, 2001) it follows that the positive 2D Lyapunov system (21) is reachable at the point (h,k) if and only if the matrix

$$\bar{R}_{hk} = \big[\bar{M}_0,\, \bar{M}_1^1, \dots, \bar{M}_h^1, \bar{M}_1^2, \dots, \bar{M}_k^2, \bar{M}_{11}, \dots, \bar{M}_{1k},\, \bar{M}_{21}, \dots, \bar{M}_{hk}\big] \quad (33)$$

contains n^2 linearly independent monomial columns, where

$$\begin{aligned} \bar{M}_0 &= \bar{T}_{h-1,k-1}\bar{B}_0,\\ \bar{M}_i^1 &= \bar{T}_{h-i,k-1}\bar{B}_1 + \bar{T}_{h-i-1,k-1}\bar{B}_0, \quad i = 1,\dots,h\\ \bar{M}_j^2 &= \bar{T}_{h-1,k-j}\bar{B}_2 + \bar{T}_{h-i,k-j-1}\bar{B}_0, \quad j = 1,\dots,k\\ \bar{M}_{i,j} &= \bar{T}_{h-i-1,k-1-1}\bar{B}_0 + \bar{T}_{h-i,k-j-1}\bar{B}_1\\ &\quad + \bar{T}_{h-i-1,k-j}\bar{B}_2, \quad i = 1,\dots,h,\ j = 1,\dots,k \end{aligned} \quad (34)$$

and $\bar{T}_{i,j}$ is the transition matrix defined in (25).

In Case (a), taking into account the assumptions, from (33), (34) and (25) we obtain

$$\begin{aligned} \bar{T}_{i,j} &= T_{i,j} \otimes I_n, \quad \bar{M}_{i,j} = M_{i,j} \otimes I_n,\\ \bar{M}_v^z &= M_v^z \otimes I_n, \quad \bar{R}_{h,k} = R_{h,k} \otimes I_n. \end{aligned}$$

Therefore, in this case, (33) contains n^2 linearly independent monomial columns if and only if (30) contains n linearly independent monomial columns.

In Case (b), from (34) we have

$$\begin{aligned} \bar{M}_h^1 &= B_1 \otimes I_n,\ \bar{M}_k^2 = B_2 \otimes I_n,\\ \bar{M}_{h-1,k-1} &= B_0 \otimes I_n + \bar{A}_2(B_1 \otimes I_n) + \bar{A}_1(B_2 \otimes I_n) \end{aligned}$$

so if the matrix $[B_1 \quad B_2]$ for $B_1 \neq 0, B_2 \neq 0$ (B_0 for $B_1 = B_2 = 0$) contains n linearly independent monomial columns, then $\bar{R}_{h,k}$ contains n^2 linearly independent monomial columns and the system is reachable. If the matrix $[B_1 \quad B_2]$ for $B_1 \neq 0, B_2 \neq 0$ (B_0 for $B_1 = B_2 = 0$) contains $r < n$ linearly independent monomial columns, then from (34) it follows that each of the matrices $\bar{M}_0, \dots, \bar{M}_{hk}$ for $h,k > 2$ contains no more than rn linearly independent monomial columns, which are linearly dependent with monomial columns of the matrix $[B_1 \quad B_2]$ for $B_1 \neq 0, B_2 \neq 0$ (B_0 for $B_1 = B_2 = 0$), and therefore the system is not reachable. ■

Remark 1. The positive 2D Lyapunov system (21) is reachable at a point (h,k), $h,k = 2$ if and only if B_0 contains n linearly independent monomial columns.

8.2. Observability

Definition 14. The positive 2D Lyapunov system (21) is called *observable* at a point $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if $X_{00} \in \mathbb{R}_+^{n\times n}$ can be uniquely determined from the knowledge of the output $Y_{i,j}$ caused by the nonzero boundary conditions in the form $X_{00} \neq 0$ and $X_{0j} = 0,\ 1 \leq j \leq k$, $X_{i0} = 0$, $1 < i < h$ and $U_{i,j} = 0,\ (i,j) \in H_{hk}$.

Theorem 8. *The positive 2D Lyapunov system (21) is observable at a point (h,k) if, and only if,* *(a) For A_l^1 satisfying the condition $XA_l^1 = A_l^1X$, i.e., $A_l^1 = a_lI_n$ and $a_l \in \mathbb{R}$, $l = 0, 1, 2$, the matrix*

$$O_{hk} = \begin{bmatrix} C\,\widehat{A}_0 \\ CT_{01}\,\widehat{A}_0 \\ \vdots \\ CT_{0,k-1}\,\widehat{A}_0 \\ CT_{10}\,\widehat{A}_0 \\ \vdots \\ CT_{h-1,k-1}\,\widehat{A}_0 \end{bmatrix},$$

$$\widehat{A}_0 = A_0^0 + A_0^1$$

contains n linearly independent monomial rows, where $T_{i,j}$ is the transition matrix defined in (32).

(b) For $A_l \neq a_lI_n$ and $a_l \in \mathbb{R}$, $l = 0, 1, 2$, if and only if the matrix $\bar{C}\bar{A}_0$ contains n^2 linearly independent monomial rows.

Proof. From Lemma 4 and (Kaczorek, 2001) it follows that the positive 2D Lyapunov system (21) is observable at a point (h, k) if and only if the matrix

$$\bar{O}_{hk} = \begin{bmatrix} \bar{C}\bar{A}_0 \\ \bar{C}\bar{T}_{01}\bar{A}_0 \\ \vdots \\ \bar{C}\bar{T}_{0,k-1}\bar{A}_0 \\ \bar{C}\bar{T}_{10}\bar{A}_0 \\ \vdots \\ \bar{C}\bar{T}_{h-1,k-1}\bar{A}_0 \end{bmatrix} \tag{35}$$

contains n^2 linearly independent monomial columns, where $\bar{T}_{i,j}$ is the transition matrix defined in (25).

In Case (a), taking into account the assumptions, from (36), (8) and the fact that $\bar{C} = C \otimes I_n$ we obtain

$$\bar{T}_{i,j} = T_{i,j} \otimes I_n, \quad \bar{O}_{h,k} = O_{h,k} \otimes I_n.$$

Therefore, in this case, (36) contains n^2 linearly independent monomial columns if and only if (35) contains n linearly independent monomial columns.

In Case (b), if the matrix $\bar{C}\bar{A}_0$ contains n^2 linearly independent monomial columns, then $\bar{O}_{h,k}$ contains n^2 linearly independent monomial columns and the system is observable. If the matrix $\bar{C}\bar{A}_0$ contains $r < n^2$ linearly independent monomial columns, then it follows that each of the matrices $\bar{C}\bar{T}_{01}\bar{A}_0, \dots, \bar{C}\bar{T}_{h-1,k-1}\bar{A}_0$ contains no more than r linearly independent monomial columns which are linearly dependent with monomial columns of the matrix $\bar{C}\bar{A}_0$. Therefore the system is not observable. ■

9. Examples

Example 1. Consider the 2D system described by the model (4) with the matrices

$$\left[\begin{array}{c|c} A_{11}^0 & A_{12}^0 \\ \hline A_{21}^0 & A_{22}^0 \end{array}\right] = \left[\begin{array}{cc|c} 0.4 & 0 & 0.1 \\ 0 & 0.5 & 0 \\ \hline 0 & 0.1 & 0.1 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{11}^1 & A_{12}^1 \\ \hline A_{21}^1 & A_{22}^1 \end{array}\right] = \left[\begin{array}{cc|c} 0.1 & 0 & 0 \\ 0 & 0.2 & 0.1 \\ \hline 0.5 & 0 & 0.2 \end{array}\right],$$

$$\left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

$$\left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right],$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$n_1 = 2, \quad n_2 = 1, \quad n = n_1 + n_2 = 3.$$

The system (37) is positive because A_{11}^0, A_{22}^0, A_{11}^1, A_{22}^1 are Metzler matrices satisfying the conditions

$$\begin{aligned} &a_{11}^{011} + a_{11}^{111} = 0.5 \geq 0, \quad a_{11}^{011} + a_{22}^{111} = 0.6 \geq 0, \\ &a_{22}^{011} + a_{11}^{111} = 0.6 \geq 0, \quad a_{22}^{011} + a_{22}^{111} = 0.7 \geq 0, \\ &a_{11}^{022} + a_{11}^{111} = 0.2 \geq 0, \quad a_{11}^{022} + a_{22}^{111} = 0.3 \geq 0, \\ &a_{11}^{011} + a_{11}^{122} = 0.6 \geq 0, \quad a_{22}^{011} + a_{22}^{122} = 0.7 \geq 0, \\ &a_{11}^{022} + a_{11}^{122} = 0.3 \geq 0, \end{aligned}$$

and A_{12}^0, A_{21}^0, A_{12}^1, A_{21}^1, B_1, B_2, C_1, C_2, D have non-negative entries.

Taking into account that the matrix

$$\begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} \left(\begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} \right)$$

has eigenvalues $\lambda_1 = 0.4, \lambda_2 = 0.1, \lambda_3 = 0.5$ ($\mu_1 = 0.2, \mu_2 = 0.2, \mu_3 = 0.1$), we obtain

$$\begin{aligned} &(\lambda_1 + \beta_1) = 0.6, \ (\lambda_1 + \beta_2) = 0.6, \ (\lambda_1 + \beta_3) = 0.5, \\ &(\lambda_2 + \beta_1) = 0.3, \ (\lambda_2 + \beta_2) = 0.3, \ (\lambda_2 + \beta_3) = 0.2, \\ &(\lambda_3 + \beta_1) = 0.7, \ (\lambda_3 + \beta_2) = 0.7, \ (\lambda_3 + \beta_3) = 0.6. \end{aligned}$$

Therefore, the system (37) is asymptotically stable, since all the sums have moduli less than one.

The system (37) is reachable at the point (h, k), $h, k > 0$ since the matrix

$$\left[\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

contains $n = 3$ linearly independent monomial columns.

The system (37) is observable at the point (h,k), $h,k>0$ since the matrix

$$C = [\, C_1 \mid C_2 \,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

contains $n = 3$ linearly independent monomial rows. ♦

Example 2. Consider the 2D system described by the model (21) with the matrices

$$A_0^0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad A_0^1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_1^0 = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_1^1 = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.1 \end{bmatrix},$$
$$A_2^0 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad n = 2. \tag{37}$$

The system (38) is positive because A_0^0, A_0^1, A_1^0, A_1^1, A_2^0, A_2^1 are Metzler matrices satisfying the conditions

$$\begin{aligned}
&a_{11}^{00} + a_{11}^{01} = 0.20 \ge 0, \quad a_{11}^{00} + a_{22}^{01} = 0.10 \ge 0,\\
&a_{22}^{00} + a_{11}^{01} = 0.30 \ge 0, \quad a_{22}^{00} + a_{22}^{01} = 0.20 \ge 0,\\
&a_{11}^{10} + a_{11}^{11} = 0.25 \ge 0, \quad a_{11}^{10} + a_{22}^{11} = 0.25 \ge 0,\\
&a_{22}^{10} + a_{11}^{11} = 0.20 \ge 0, \quad a_{22}^{10} + a_{22}^{11} = 0.20 \ge 0,\\
&a_{11}^{20} + a_{11}^{21} = 0.20 \ge 0, \quad a_{11}^{20} + a_{22}^{21} = 0.30 \ge 0,\\
&a_{22}^{20} + a_{11}^{21} = 0.0 \ge 0, \quad a_{22}^{20} + a_{22}^{21} = 0.10 \ge 0,
\end{aligned}$$

and B_0, B_1, B_2, C, D have nonnegative entries.

Taking into account that the matrix

$$\begin{bmatrix} A_1^0 + A_2^0 & A_0^0 \\ I_2 & 0_2 \end{bmatrix} \left(\begin{bmatrix} A_1^1 + A_2^1 & A_0^1 \\ I_2 & 0_2 \end{bmatrix} \right)$$

has eigenvalues $\lambda_1 = 0.5364$, $\lambda_2 = -0.1864$, $\lambda_3 = 0.5$, $\lambda_4 = -0.4$ ($\mu_1 = 0.3702$, $\mu_2 = -0.2702$, $\mu_3 = 0$, $\mu_4 = 0.2$), we obtain

$$\begin{aligned}
&(\lambda_1 + \beta_1) = 0.9066, && (\lambda_1 + \beta_2) = 0.2662,\\
&(\lambda_1 + \beta_3) = 0.5364, && (\lambda_1 + \beta_4) = 0.7364,\\
&(\lambda_2 + \beta_1) = 0.1838, && (\lambda_2 + \beta_2) = -0.4566,\\
&(\lambda_2 + \beta_3) = -0.1864, && (\lambda_2 + \beta_4) = 0.0136,\\
&(\lambda_3 + \beta_1) = 0.65, && (\lambda_3 + \beta_2) = 0.65,\\
&(\lambda_3 + \beta_3) = 0.5, && (\lambda_3 + \beta_4) = 0.3\\
&(\lambda_4 + \beta_1) = 0.8702, && (\lambda_4 + \beta_2) = 0.2298,\\
&(\lambda_4 + \beta_3) = -0.4, && (\lambda_4 + \beta_4) = -0.2.
\end{aligned}$$

Therefore, the system (38) is asymptotically stable, since all the sums have moduli less than one.

The system (38) is reachable at the point (h,k), $h,k>2$ since the matrix

$$[\, B_1 \;\; B_2 \,] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

contains $n = 2$ linearly independent monomial columns.

The system is observable at the point (h,k), $h,k>0$ since the matrix

$$\bar{C}\bar{A}_0 = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

contains $n^2 = 4$ linearly independent monomial rows. ♦

10. Concluding remarks

The notion of a positive 2D discrete-time linear Lyapunov system described by two different models have been introduced. For both the models necessary and sufficient conditions for positivity (Theorems 1 and 5), asymptotic stability (Theorems 2 and 6), reachability (Theorems 3 and 7) and observability (Theorems 4 and 8) were established. The discussion was illustrated with numerical examples. Minimum energy control and constrained controllability of 2D Lyapunov systems are open problems. So is the determination of relationships between the presented models.

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Przemysław Przyborowski was born in 1981 in Poland. He received the M.Sc. degree in electrical enginnering and the Ph.D. degree in automation and robotics from the Warsaw University of Technology in 2005 and 2008, respectively. His research interests cover systems and control theories, especially Lyapunov positive 1D and 2D systems, systems with uncertainties and nonlinearities, industrial automation. He is an author or co-author of 10 scientific papers.

Tadeusz Kaczorek was born in 1932 in Poland. He received the M.Sc., Ph.D. and D.Sc. degrees from the Faculty of Electrical Engineering of the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the period 1968–69 he was the dean of the Faculty of Electrical Engineering, and in the period 1970–73 he was the deputy rector of the Warsaw University of Technology. Since 1974 he has been a full professor at the Warsaw University of Technology. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the period 1988–1991 he was the director of the Research Centre of the Polish Academy of Sciences in Rome. In June 1999 he was elected a full member of the Academy of Engineering in Poland. In 2004 he was elected a honorary member of the Hungarian Academy of Sciences. He has been awarded an honorary doctorate by several universities. His research interests cover systems theory and control theory, especially singular multidimensional systems, positive multidimensional systems, and singular positive 1D and 2D systems. He has published 21 books (six in English) and over 850 scientific papers.

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