

FAULT TOLERANT CONTROL OF SWITCHED NONLINEAR SYSTEMS WITH TIME DELAY UNDER ASYNCHRONOUS SWITCHING

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This paper investigates the problem of fault tolerant control of a class of uncertain switched nonlinear systems with time delay under asynchronous switching. The systems under consideration suffer from delayed switchings of the controller. First, a fault tolerant controller is proposed to guarantee exponential stability of the switched systems with time delay. The dwell time approach is utilized for stability analysis and controller design. Then the proposed approach is extended to take into account switched time delay systems with Lipschitz nonlinearities and structured uncertainties. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: time delay, fault tolerant control, switched nonlinear systems, asynchronous switching.

1. Introduction

Switched systems belong to a special class of hybrid control systems that comprises a collection of subsystems together with a switching rule which specifies the switching among the subsystems. Many practical systems are inherently multimodal in the sense that several dynamical systems are required to describe their behavior, which may depend on various environmental factors. Besides, switched systems are widely applied in many fields, including mechanical systems, automotive industry, aircraft and air traffic control, and many other domains (Varaiya, 1993; Wang and Brockett, 1997; Tomlin *et al.*, 1998).

During the last decades there have been many studies on stability analysis and the design of stabilizing feedback controllers for switched systems. The interest in this direction is reflected by numerous works (Sun, 2004; 2006; Cheng *et al.*, 2005; Liberzon, 2003; Lin and Antsaklis, 2009). As an important analytic tool, the multiple Lyapunov function approach has been employed to analyze the stability of switched systems, which has been shown to be

very efficient (Zhai *et al.*, 2007; Hespanha, 2004; Hespanha *et al.*, 2005). Based on the dwell time method, stability analysis and stabilization for switched systems have also been investigated (De Persis *et al.*, 2002; Wang and Zhao, 2007; Sun *et al.*, 2006a; De Persis *et al.*, 2003).

The time delay phenomenon is very common in many kinds of engineering systems, for instance, long-distance transportation systems, hydraulic pressure systems, networked control systems and so on, so time delay systems have also received increased attention in the control community (Guo and Gao, 2007; Guan and Gao, 2007). Many valuable results have been obtained for systems of this type (Zhang *et al.*, 2007a; Gao *et al.*, 2008; Xiang and Wang, 2009a; Sun *et al.*, 2006b; Zhang *et al.*, 2007b). On the other hand, actuators may be subjected to failures in a real environment. Therefore, it is of practical interest to investigate a control system which can tolerate faults of actuators. Several approaches to the design of reliable controllers have been proposed (Lien *et al.*, 2008; Yao and Wang, 2006; Abootalebi *et al.*, 2005; Liu *et al.*,

1998; Yu, 2005). A reliable controller is designed for switched nonlinear systems using the multiple Lyapunov function approach by Wang *et al.* (2007).

However, there inevitably exists asynchronous switching between the controller and the system in actual operation, which deteriorates the performance of systems. Therefore, it is important to investigate the problem of the stabilization of switched systems under asynchronous switching (Xie and Wang, 2005; Xie *et al.*, 2001; Ji *et al.*, 2007; Hetel *et al.*, 2007; Mhaskar *et al.*, 2008; Xiang and Wang, 2009b).

In this paper, we are interested in the problem of fault tolerant control for a class of uncertain nonlinear switched systems with time delay and actuator failures under asynchronous switching. The remainder of the paper is organized as follows. In Section 2, problem formulation and some necessary lemmas are given. In Section 3, based on the dwell time approach and the linear matrix inequality (LMI) technique, we first consider the design of a fault tolerant controller and a switching signal for a switched system with time delay under asynchronous switching. Sufficient conditions for the existence of the controller are obtained in terms of a set of LMIs. Then the design approach to the controller for a switched nonlinear system with time delay under asynchronous switching is presented. A numerical example is given to illustrate the effectiveness of the proposed design approach in Section 4. Concluding remarks are given in Section 5.

Notation. Throughout this paper, the superscript ‘*T*’ denotes the transpose, $\|\cdot\|$ denotes the Euclidean norm. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P , respectively, I is an identity matrix of appropriate dimensions. The asterisk ‘*’ in a matrix is used to denote a term that is induced by symmetry. The set of positive integers is represented by \mathbb{Z}^+ .

2. System description and preliminaries

Let us consider the following switched system with time delay and an actuator failure:

$$\dot{x}(t) = \hat{A}_{\sigma(t)}x(t) + \hat{A}_{d\sigma(t)}x(t-d) + B_{\sigma(t)}u^f(t) + D_{\sigma(t)}f_{\sigma(t)}(x(t), t), \quad (1)$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0], \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u^f(t) \in \mathbb{R}^l$ is the input of an actuator fault, d denotes the state delay, $\phi(t)$ is a continuous vector-valued function. The function $\sigma(t) : [t_0, \infty) \rightarrow \underline{N} = \{1, 2, \dots, N\}$ is the system switching signal, and N denotes the number of the subsystems. The switching signal $\sigma(t)$ discussed in this paper is time-dependent, i.e., $\sigma(t) : \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \dots\}$, where t_0 is the initial time, and t_k denotes the k -th switching instant. \hat{A}_i, \hat{A}_{di} for $i \in \underline{N}$ are uncertain real-valued

matrices with appropriate dimensions which satisfy

$$\hat{A}_i = A_i + H_i F_i(t) E_{1i}, \quad \hat{A}_{di} = A_{di} + H_i F_i(t) E_{di}, \quad (3)$$

where $A_i, A_{di}, H_i, E_{1i}, E_{di}$ are known real constant matrices with proper dimensions imposing the structure of the uncertainties. Here $F_i(t)$ for $i \in \underline{N}$ are unknown time-varying matrices which satisfy

$$F_i^T(t) F_i(t) \leq I, \quad (4)$$

D_i and B_i for $i \in \underline{N}$ are known real constant matrices, and $f_i(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ for $i \in \underline{N}$ are unknown nonlinear functions satisfying the following Lipschitz conditions:

$$\|f_i(x(t), t)\| \leq \|U_i x(t)\|, \quad (5)$$

where U_i are known real constant matrices.

However, there inevitably exists asynchronous switching between the controller and the system in actual operation. Suppose that the i -th subsystem is activated at the switching instant t_{k-1} , the j -th subsystem is activated at the switching instant t_k , and the corresponding switching controller is activated at the switching instants $t_{k-1} + \Delta_{k-1}$ and $t_k + \Delta_k$, respectively. The case that the switching instants of the controller experience delays with respect to those of the system can be shown as in Fig. 1. There we can see that controller K_i correspon-

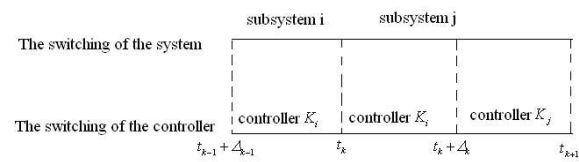


Fig. 1. Diagram of asynchronous switching.

ding to the i -th subsystem operates the i -th subsystem in $[t_{k-1} + \Delta_{k-1}, t_k)$, and operates the j -th subsystem in $[t_k, t_k + \Delta_k)$.

Denoting by $\sigma'(t)$ the switching signal of the controller, the corresponding switching instants can be written as

$$t_1 + \Delta_1, t_2 + \Delta_2, \dots, t_k + \Delta_k, \dots, k \in \mathbb{Z}^+,$$

where Δ_k ($|\Delta_k| < d$) represents the period that the switching instant of the controller lags behind the one of the system, and the period is said to be mismatched.

Remark 1. The mismatched period

$$\Delta_k < \inf_{k \geq 0} (t_{k+1} - t_k)$$

guarantees that there always exists a period $[t_{k-1} + \Delta_{k-1}, t_k)$. This period is said to be matched in what follows.

The input of an actuator fault is described as

$$u^f(t) = M_{\sigma'(t)}u(t), \quad (6)$$

where M_i for $i \in \underline{N}$ are actuator fault matrices,

$$M_i = \text{diag}\{m_{i1}, m_{i2}, \dots, m_{il}\}$$

$$0 \leq \underline{m}_{ik} \leq m_{ik} \leq \overline{m}_{ik}, \quad \overline{m}_{ik} \geq 1, k = 1, 2, \dots, l. \quad (7)$$

For simplicity, we introduce the following notation:

$$M_{i0} = \text{diag}\{\tilde{m}_{i1}, \tilde{m}_{i2}, \dots, \tilde{m}_{il}\}, \quad (8)$$

$$J_i = \text{diag}\{j_{i1}, j_{i2}, \dots, j_{il}\}, \quad (9)$$

$$L_i = \text{diag}\{l_{i1}, l_{i2}, \dots, l_{il}\}, \quad (10)$$

where

$$\begin{aligned} \tilde{m}_{ik} &= \frac{1}{2}(\overline{m}_{ik} + \underline{m}_{ik}), \\ j_{ik} &= \frac{\overline{m}_{ik} - \underline{m}_{ik}}{\overline{m}_{ik} + \underline{m}_{ik}}, \\ l_{ik} &= \frac{m_{ik} - \tilde{m}_{ik}}{\tilde{m}_{ik}}. \end{aligned}$$

By (8)–(10), we have

$$M_i = M_{i0}(I + L_i), \quad |L_i| \leq J_i \leq I, \quad (11)$$

where

$$|L_i| = \text{diag}\{|l_{i1}|, |l_{i2}|, \dots, |l_{il}|\}.$$

Remark 2. Note that $m_{ik} = 1$ means normal operation of the k -th actuator signal of the i -th subsystem. When $m_{ik} = 0$, it covers the case of the complete failure of the k -th actuator signal of the i -th subsystem. When $\underline{m}_{ik} > 0$ and $m_{ik} \neq 1$, it corresponds to the case of a partial failure of the k -th actuator signal of the i -th subsystem. The system (1)–(2) without uncertainties can be described as

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d) \\ &\quad + B_{\sigma(t)}u^f(t) + D_{\sigma(t)}f_{\sigma(t)}(x(t), t), \end{aligned} \quad (12)$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0]. \quad (13)$$

The system (12)–(13) without nonlinear terms can be written as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d) + B_{\sigma(t)}u^f(t), \quad (14)$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0]. \quad (15)$$

Definition 1. If there exists a switching signal $\sigma(t)$, such that the trajectory of the system (1)–(2) satisfies $\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\beta(t-t_0)}$, where $\alpha \geq 1$, $\beta > 0$, $t \geq t_0$, then the system (1)–(2) is said to be *exponentially stable*.

The following lemmas play an important role in our further developments.

Lemma 1. (Halanay, 1966) *Let $r \geq 0$, $a > b > 0$. If there exists a real-value continuous function $u(t) \geq 0$, $t \geq t_0$ such that the differential inequality*

$$\frac{du(t)}{dt} \leq -au(t) + b \sup_{t-r \leq \theta \leq t} u(\theta), \quad t \geq t_0$$

holds, then

$$u(t) \leq \sup_{-r \leq \theta \leq 0} u(t_0 + \theta) e^{-\mu(t-t_0)}, \quad t \geq t_0,$$

where $\mu > 0$, and

$$\mu - a + be^{\mu r} = 0$$

is satisfied.

Lemma 2. (Xiang and Wang, 2009a) *For matrices X, Y with appropriate dimensions and a matrix $Q > 0$, we have*

$$X^T Y + Y^T X \leq X^T Q X + Y^T Q^{-1} Y.$$

Lemma 3. (Petersen, 1987) *For matrices R_1, R_2 with appropriate dimensions, there exists a positive scalar $\beta > 0$ such that*

$$R_1 \Sigma(t) R_2 + R_2^T \Sigma^T(t) R_1^T \leq \beta R_1 U R_1^T + \beta^{-1} R_2^T U R_2,$$

where $\Sigma(t)$ is a time-varying diagonal matrix, U is a known real-value matrix satisfying $|\Sigma(t)| \leq U$.

Lemma 4. (Xiang and Wang, 2009a) *Let U, V, W and X be real matrices of appropriate dimensions with X satisfying $X = X^T$. Then for all $V^T V \leq I$ we have*

$$X + UVW + W^T V^T U^T < 0$$

if and only if there exists a scalar $\varepsilon > 0$ such that

$$X + \varepsilon U U^T + \varepsilon^{-1} W^T W < 0.$$

Lemma 5. (Boyd, 1994, Schur Complement) *For a given matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

with $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, the following condition is equivalent:

(1) $S < 0$

(2) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

The objective of this paper is to design a fault tolerant controller such that the system (1)–(2) under asynchronous switching is robust exponentially stable.

3. Main results

To obtain our main results, consider the system (12)–(13) with the asynchronous switching controller $u(t) = K_{\sigma'(t)}x(t)$. The corresponding closed-loop system is given by

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}M_{\sigma'(t)}K_{\sigma'(t)})x(t) + A_{d\sigma(t)}x(t-d) + D_{\sigma(t)}f_{\sigma(t)}(x(t), t), \quad (16)$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0]. \quad (17)$$

Lemma 6. Consider the system (12)–(13), for given positive scalars $\alpha, \eta > 0$, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$ and matrices Y_i for fault matrix M_i , such that for $i, j \in \underline{N}$

$$\begin{bmatrix} \Xi_i & A_{di}X_i & D_i & X_iU_i^T \\ * & -X_i & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \quad (18)$$

$$\begin{bmatrix} \Xi_{ij} & P_{ij}A_{dj} & P_{ij}D_j \\ * & -P_{ij} & 0 \\ * & * & -I \end{bmatrix} < 0 \quad (19)$$

and the dwell time satisfies $\inf_{k \geq 0} (t_{k+1} - t_k) \geq T$. Then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (20)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned} \Xi_i &= (A_iX_i + B_iM_iY_i)^T + A_iX_i + B_iM_iY_i \\ &\quad + (1 + \alpha)X_i, \\ \Xi_{ij} &= (A_j + B_jM_jY_jX_i^{-1})^T P_{ij} + P_{ij}(A_j + B_jM_jY_jX_i^{-1}) \\ &\quad + (1 + \eta)P_{ij} + U_j^T U_j, \end{aligned}$$

$$\begin{aligned} T &> 2d + \frac{\ln \rho_1 \rho_2}{\mu}, \\ \rho_1 &= \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\}, \\ \rho_2 &= \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\}, \end{aligned}$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Proof. See Appendix. ■

The following theorem presents sufficient conditions for the existence of a fault tolerant controller for the system (1)–(2) under asynchronous switching.

Theorem 1. Consider the system (1)–(2). For given positive scalars $\alpha, \eta > 0$, if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, positive scalars

$\varepsilon_i, \beta_i, \zeta_i, \theta_i$, and matrices Y_i , such that for $i, j \in \underline{N}$

$$\begin{bmatrix} \Theta_i & A_{di}X_i & D_i & X_iU_i^T & Y_iM_{i0}J_i^{1/2} & X_iE_{1i}^T \\ * & -X_i & 0 & 0 & 0 & X_iE_{di}^T \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & * & -\beta_i I \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} \Theta_{ij} & P_{ij}A_{dj} & P_{ij}D_j & \zeta_j X_i^{-1} Y_i^T M_{i0} J_i^{1/2} \\ * & -P_{ij} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\zeta_j I \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} P_{ij}B_j J_i^{1/2} & \theta_j E_{1j}^T & P_{ij}H_j \\ 0 & \theta_j E_{dj}^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\zeta_j I & 0 & 0 \\ * & -\theta_j I & 0 \\ * & * & -\theta_j I \end{bmatrix} < 0 \quad (22)$$

and the dwell time satisfies $\inf_{k \geq 0} (t_{k+1} - t_k) \geq T$, then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \quad (23)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned} \Theta_i &= (A_iX_i + B_iM_{i0}Y_i)^T + A_iX_i + B_iM_{i0}Y_i \\ &\quad + (1 + \alpha)X_i + \beta_i H_{1i} H_{1i}^T + \varepsilon_i B_i J_i B_i^T, \\ \Theta_{ij} &= (A_j + B_jM_{i0}Y_iX_i^{-1})^T P_{ij} \\ &\quad + P_{ij}(A_j + B_jM_{i0}Y_iX_i^{-1}) \\ &\quad + (1 + \eta)P_{ij} + U_j^T U_j, \end{aligned}$$

$$\begin{aligned} T &> 2d + \frac{\ln \rho_1 \rho_2}{\mu}, \\ \rho_1 &= \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\}, \\ \rho_2 &= \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\}, \end{aligned}$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Proof. Consider the system (1)–(2) with the controller $u(t) = K_{\sigma'(t)}x(t)$. The corresponding closed-loop system is given by

$$\dot{x}(t) = (\hat{A}_{\sigma(t)} + B_{\sigma(t)}M_{\sigma'(t)}K_{\sigma'(t)})x(t) + \hat{A}_{d\sigma(t)}x(t-d) + D_{\sigma(t)}f_{\sigma(t)}(x(t), t), \quad (24)$$

$$x(t) = \phi(t), \quad t \in [t_0 - d, t_0]. \quad (25)$$

Write

$$T_i = \begin{bmatrix} \Lambda_i & \hat{A}_{di}X_i & D_i & X_iU_i^T & Y_i^T M_{i0}J_i^{1/2} \\ * & -X_i & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix}, \quad (26)$$

where

$$\Lambda_i = (\hat{A}_i X_i + B_i M_{i0} Y_i)^T + \hat{A}_i X_i + B_i M_{i0} Y_i + (1 + \alpha)X_i + \varepsilon_i B_i J_i B_i^T.$$

Substituting (11) to (26) and using Lemma 4, it is easy to see that (21) is equivalent to $T_i < 0$.

Write

$$Z_{ij} = \begin{bmatrix} \Lambda_{ij} & P_{ij}\hat{A}_{dj} & P_{ij}D_j \\ * & -P_{ij} & 0 \\ * & * & -I \\ * & * & * \\ * & * & * \\ \zeta_j X_i^{-1} Y_i^T M_{i0} J_i^{1/2} & P_{ij} B_j J_i^{1/2} \\ 0 & 0 \\ 0 & 0 \\ -\zeta_j I & 0 \\ * & -\zeta_j I \end{bmatrix},$$

where

$$\Lambda_{ij} = (\hat{A}_j + B_j M_{i0} Y_i X_i^{-1})^T P_{ij} + P_{ij}(\hat{A}_j + B_j M_{i0} Y_i X_i^{-1}) + (1 + \eta)P_{ij} + U_j^T U_j.$$

Following a similar proof line, we have $Z_{ij} < 0$ from (22). From Lemma 6 we conclude that Theorem 1 holds. The proof is completed. ■

Remark 3. Note that the matrix inequalities (21) and (22) are mutually constrained. Therefore, we can first solve the linear matrix inequality (21) to obtain matrices X_i and Y_i . Then we solve (22) by substituting X_i and Y_i into (22). By adjusting the parameter α, η appropriately, feasible solutions X_i, Y_i , and P_{ij} can be found such that the matrix inequalities (21) and (22) hold.

From Theorem 1, we can easily obtain the following results.

Corollary 1. Consider the system (14)–(15). For given positive scalars α, η , if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, matrices Y_i and positive scalars $\varepsilon_i > 0, \zeta_i > 0$, such that for $i, j \in \underline{N}$

$$\begin{bmatrix} \Gamma_i & A_{di}X_i & Y_i^T M_{i0} J_i^{1/2} \\ * & -X_i & 0 \\ * & * & -\varepsilon_i I \end{bmatrix} < 0, \quad (27)$$

$$\begin{bmatrix} \Gamma_{ij} & P_{ij}A_{dj} & \zeta_j X_i^{-1} Y_i^T M_{i0} J_i^{1/2} & P_{ij}B_j J_i^{1/2} \\ * & -P_{ij} & 0 & 0 \\ * & * & -\zeta_j I & 0 \\ * & * & * & -\zeta_j I \end{bmatrix} < 0 \quad (28)$$

and the dwell time satisfies $\inf_{k \geq 0} (t_{k+1} - t_k) \geq T$, then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_i X_i^{-1}, \quad (29)$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\Gamma_i = (A_i X_i + B_i M_{i0} Y_i)^T + A_i X_i + B_i M_{i0} Y_i + (1 + \alpha)X_i + \varepsilon_i B_i J_i B_i^T,$$

$$\Gamma_{ij} = (A_j + B_j M_{i0} Y_i X_i^{-1})^T P_{ij} + P_{ij}(A_j + B_j M_{i0} Y_i X_i^{-1}) + (1 + \eta)P_{ij},$$

$$T > 2d + \frac{\ln \rho_1 \rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i, j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\},$$

$$\rho_2 = \max_{\substack{i, j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\},$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

Corollary 2. Consider the system (12)–(13). For given positive scalars α, η , if there exist symmetric positive definite matrices $X_i > 0, P_{ij} > 0$, matrices Y_i and positive scalar $\varepsilon_i > 0, \zeta_i > 0$, such that for $i, j \in \underline{N}$

$$\begin{bmatrix} \Sigma_i & A_{di}X_i & D_i & X_iU_i^T & Y_i^T M_{i0} J_i^{1/2} \\ * & -X_i & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix} < 0 \quad (30)$$

$$\begin{bmatrix} \Sigma_{ij} & P_{ij}A_{dj} & P_{ij}D_j & \zeta_j X_i^{-1} Y_i^T M_{i0} J_i^{1/2} \\ & -P_{ij} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\zeta_j I \\ * & * & * & * \end{bmatrix}$$

$$\begin{bmatrix} P_{ij}B_jJ_i^{1/2} \\ 0 \\ 0 \\ 0 \\ -\zeta_j I \end{bmatrix} < 0 \tag{31}$$

and the dwell time satisfies $\inf_{k \geq 0} (t_{k+1} - t_k) \geq T$, then there exists a controller

$$u(t) = K_{\sigma'(t)}x(t), \quad K_i = Y_iX_i^{-1}, \tag{32}$$

which can guarantee that the closed-loop system is exponentially stable, where

$$\begin{aligned} \Sigma_i &= (A_iX_i + B_iM_{i0}Y_i)^T + A_iX_i + B_iM_{i0}Y_i \\ &\quad + (1 + \alpha)X_i + \varepsilon_iB_iJ_iB_i^T, \\ \Sigma_{ij} &= (A_j + B_jM_{i0}Y_iX_i^{-1})^T P_{ij} \\ &\quad + P_{ij}(A_j + B_jM_{i0}Y_iX_i^{-1}) \\ &\quad + (1 + \eta)P_{ij} + U_j^T U_j, \end{aligned}$$

$$T > 2d + \frac{\ln \rho_1 \rho_2}{\mu},$$

$$\rho_1 = \max_{\substack{i,j \in \mathcal{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(X_j^{-1})}{\lambda_{\min}(P_{ij})} \right\},$$

$$\rho_2 = \max_{\substack{i,j \in \mathcal{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(X_i^{-1})} \right\},$$

μ satisfies $\mu + e^{\mu d} = 1 + \min\{\alpha, \eta\}$.

4. Numerical example

In this section, an example is given to illustrate the effectiveness of the proposed method. Consider the system (1)–(2) with the following parameters:

$$\begin{aligned} 2A_1 &= \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, & A_2 &= \begin{bmatrix} -0.2 & 0 \\ 0 & -0.3 \end{bmatrix}, \\ A_{d1} &= \begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}, & A_{d2} &= \begin{bmatrix} -0.2 & 0.3 \\ 0 & -0.1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -8 & 0 \\ 0 & 7 \end{bmatrix}, & B_2 &= \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} 0.3 & -0.2 \\ 0 & -0.1 \end{bmatrix}, & D_2 &= \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & 0.2 \end{bmatrix}, \\ U_1 &= \begin{bmatrix} -0.1 & 0 \\ 0 & 0 \end{bmatrix}, & U_2 &= \begin{bmatrix} 0 & -0.1 \\ 0 & 0 \end{bmatrix}, \\ H_1 &= \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.3 \end{bmatrix}, & H_2 &= \begin{bmatrix} 0.4 & 0 \\ 0.2 & 0 \end{bmatrix}, \\ E_{11} &= \begin{bmatrix} 0 & 0.6 \\ 0 & 0 \end{bmatrix}, & E_{12} &= \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}, \\ E_{d1} &= \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.6 \end{bmatrix}, & E_{d2} &= \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \end{aligned}$$

$$d = 1.2,$$

$$\begin{aligned} f_1(x(t), t) &= \begin{bmatrix} 0.1 \sin x_1 \\ 0 \end{bmatrix}, \\ f_2(x(t), t) &= \begin{bmatrix} 0 \\ 0.1 \sin x_2 \end{bmatrix}. \end{aligned}$$

The fault matrices are as follows:

$$\begin{aligned} 0.1 &\leq m_{11} \leq 0.5, \\ 0.2 &\leq m_{12} \leq 0.8, \\ 0.2 &\leq m_{21} \leq 0.4, \\ 0.3 &\leq m_{22} \leq 0.9., \end{aligned}$$

that is,

$$\begin{aligned} 2M_{10} &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}, & M_{20} &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.6 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} 0.67 & 0 \\ 0 & 0.6 \end{bmatrix}, & J_2 &= \begin{bmatrix} 0.33 & 0 \\ 0 & 0.5 \end{bmatrix}. \end{aligned}$$

Choosing $\alpha = 3, \eta = 2$, by solving the LMIs in Theorem 1, we have

$$\begin{aligned} K_1 &= \begin{bmatrix} 6.6186 & 0.8014 \\ -0.4395 & -3.5108 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} 5.8192 & 0.9082 \\ -0.6788 & -4.8117 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} X_1 &= \begin{bmatrix} 1.4270 & -0.1871 \\ -0.1871 & 1.3903 \end{bmatrix}, \\ X_2 &= \begin{bmatrix} 1.8435 & -0.2754 \\ -0.2754 & 1.5919 \end{bmatrix}, \\ P_{12} &= \begin{bmatrix} 1.8322 & 0.3021 \\ 0.3021 & 1.4479 \end{bmatrix}, \\ P_{21} &= \begin{bmatrix} 3.9085 & 0.3765 \\ 0.3765 & 2.8780 \end{bmatrix}, \end{aligned}$$

$\rho_1 = 0.6390, \rho_2 = 8.1459, T > 4.7$. Choose the switching signal as follows

$$\sigma(t) = \begin{cases} 1, & 2k\tau^* \leq t < (2k+1)\tau^*, \\ 2, & (2k+1)\tau^* \leq t < (2k+2)\tau^*, \end{cases}$$

where $k = 0, 1, 2, \dots, \tau^* = 5$.

The state response of the closed-loop system is shown in Fig. 2, where $\Delta_k = 1 (k = 1, 2)$ and the initial condition is

$$x(t) = [2 \quad -1]^T, \quad t \in [-1.2, 0].$$

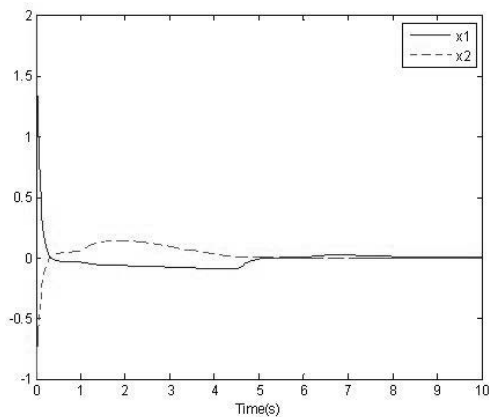


Fig. 2. State response of the closed-loop system.

5. Conclusion

This paper investigates the problem of fault tolerant control for a class of uncertain switched nonlinear systems with time delay and actuator failures under asynchronous switching. Sufficient conditions for the existence of a fault tolerant control law were derived. The proposed controller can be obtained by solving a set of LMIs. A numerical example was provided to show the effectiveness of the proposed approach.

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References

- Abotalebi, A., Hossein Nia, S. and Sheikholeslam, F. (2005). Reliable H_∞ control systems with uncertainties in all system matrices: An LMI approach, *Proceedings of the 2005 IEEE Conference on Control Applications, Toronto, Canada*, pp. 251–255.
- Boyd, S.P., Ghaoui, L.E., Feron, E. and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA.
- Cheng, D., Guo, L., Lin, Y. and Wang, Y. (2005). Stabilization of switched linear systems, *IEEE Transactions on Automatic Control* **50**(5): 661–666.
- DePersis, C., De Santis, R. and Morse, A.S. (2002). Nonlinear switched systems with state dependent dwell-time, *Proceedings of the 2002 IEEE Conference on Decision and Control, Las Vegas, NV, USA*, pp. 4419–4424.
- DePersis, C., De Santis, R. and Morse, A.S. (2003). Switched nonlinear systems with state-dependent dwell-time, *Systems and Control Letters* **50**(4): 291–302.
- Guan, H.W. and Gao, L.X. (2007). Delay-dependent robust stability and H_∞ control for jump linear system with interval time-varying delay, *Proceedings of the 26th Chinese Control Conference, Zhangjiajie, China*, pp. 609–614.
- Guo, J.F. and Gao, C.C. (2007). Output variable structure control for time-invariant linear time-delay singular system, *Journal of Systems Science and Complexity* **20**(3): 454–460.
- Gao, F.Y., Zhong, S.M. and Gao, X.Z. (2008). Delay-dependent stability of a type of linear switching systems with discrete and distributed time delays, *Applied Mathematics and Computation* **196**(1): 24–39.
- Halanay, A. (1966). *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, NY.
- Hespanha, J.P. (2004). Uniform stability of switched linear systems: Extension of LaSalle's invariance principle, *IEEE Transactions on Automatic Control* **48**(4): 470–482.
- Hespanha, J.P., Liberzon, D., Angeli, D. and Sontag, E.D. (2005). Nonlinear norm-observability notions and stability of switched systems, *IEEE Transactions on Automatic Control* **50**(2): 154–168.
- Hetel, L., Daafouz, J. and Jung, C. (2007). Stability analysis for discrete time switched systems with temporary uncertain switching signal, *Proceedings of the IEEE Conference on Decision and Control, New Orleans, LA, USA*, pp. 5623–5628.
- Ji, Z., Guo, X., Xu, S. and Wang, L. (2007). Stabilization of switched linear systems with time-varying delay in switching occurrence detection, *Circuits, Systems and Signal Processing* **26**(3): 361–367.
- Liberzon, D. (2003). *Switching in Systems and Control*, Birkhauser, Boston, MA.
- Lien, C.H., Yu, K.W., Lin, Y.F., Chung Y.J. and Chung, L.Y. (2008). Robust reliable H_∞ control for uncertain nonlinear systems via LMI approach, *Applied Mathematics and Computation* **198**(1): 453–462.
- Lin, H, Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: A survey of recent results, *IEEE Transactions on Automatic Control* **54**(2): 308–322.
- Liu, Y.Q., Wang, J.L. and Yang, G.H. (1998). Reliable control of uncertain nonlinear systems, *Automatica* **34**(7): 875–879.
- Mhaskar, P., El-Farra, N.H. and Christofides, P.D. (2008). Robust predictive control of switched systems: Satisfying uncertain schedules subject to state and control constraints, *International Journal of Adaptive Control and Signal Processing* **22**(2): 161–179.
- Petersen, I.R. (1987). A stabilization algorithm for a class of uncertain linear systems, *Systems and Control Letters* **8**(4): 351–357.
- Sun, X.M., Dimirovski, G.M., Zhao, J., Wang, W. and Cyril, S.S. (2006a). Exponential stability for switched delay systems based on average dwell time technique and Lyapunov function method, *Proceedings of the 2006 American Control Conference, Minneapolis, MN, USA*, pp. 1539–1543.
- Sun, X.M., Zhao, J. and David, J.H. (2006b). Stability and L_2 -gain analysis for switched delay systems: A delay-dependent method, *Automatica* **42**(10): 1769–1774.

- Sun, Z. (2004). A robust stabilizing law for switched linear systems, *International Journal of Control* **77**(4): 389–398.
- Sun, Z. (2006). Combined stabilizing strategies for switched linear systems, *IEEE Transactions on Automatic Control* **51**(4): 666–674.
- Tomlin, C., Pappas, G.J., Sastry, S. (1998). Conflict resolution for air traffic management: A study in multiagent hybrid systems, *IEEE Transactions on Automatic Control* **43**(4): 509–521.
- Varaiya, P. (1993). Smart cars on smart roads: Problems of control, *IEEE Transactions on Automatic Control* **38**(2): 195–207.
- Wang, W. and Brockett, R.W. (1997). Systems with finite communication bandwidth constraints—Part I: State estimation problems, *IEEE Transactions on Automatic Control* **42**(9): 1294–1299.
- Wang, R., Liu, M. and Zhao, J. (2007). Reliable H_∞ control for a class of switched nonlinear systems with actuator failures, *Nonlinear Analysis: Hybrid Systems* **1**(3): 317–325.
- Wang, R. and Zhao, J. (2007). Guaranteed cost control for a class of uncertain switched delay systems: An average dwell-time method, *Cybernetics and Systems* **38**(1): 105–122.
- Xiang, Z. and Wang, R. (2009a). Robust L_∞ reliable control for uncertain nonlinear switched systems with time delay, *Applied Mathematics and Computation* **210**(1): 202–210.
- Xiang, Z.R. and Wang, R.H. (2009b). Robust control for uncertain switched non-linear systems with time delay under asynchronous switching, *IET Control Theory and Applications* **3**(8): 1041–1050.
- Xie, G. and Wang, L. (2005). Stabilization of switched linear systems with time-delay in detection of switching signal, *Applied Mathematics and Computation* **305**(6): 277–290.
- Xie, W., Wen, C. and Li, Z. (2001). Input-to-state stabilization of switched nonlinear systems, *IEEE Transactions on Automatic Control* **46**(7): 1111–1116.
- Yao, B. and Wang, F.Z. (2006). LMI approach to reliable H_∞ control of linear systems, *Journal of Systems Engineering and Electronics* **17**(2): 381–386.
- Yu, L. (2005). An LMI approach to reliable guaranteed cost control of discrete-time systems with actuator failure, *Applied Mathematics and Computation* **162** (3): 1325–1331.
- Zhai, G., Xu, X., Lin, H. and Liu, D. (2007). Extended Lie algebraic stability analysis for switched systems with continuous-time and discrete-time subsystems, *International Journal of Applied Mathematics and Computer Science* **17**(4): 447–454, DOI: 10.2478/v10006-007-0036-x.
- Zhang, L.X., Wang, C.H. and Gao, H.J. (2007). Delay-dependent stability and stabilization of a class of linear switched time-varying delay systems, *Journal of Systems Engineering and Electronics* **18**(2): 320–326.
- Zhang, Y., Liu, X.Z. and Shen, X.M. (2007). Stability of switched systems with time delay, *Nonlinear Analysis: Hybrid Systems* **1**(1): 44–58.



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Appendix

Proof of Lemma 6. Without loss of generality, we assume the initial time $t_0 = 0$.

When $t \in [t_{k-1} + \Delta_{k-1}, t_k)$, the closed-loop system (16)–(17) can be written as

$$\dot{x}(t) = (A_i + B_i M_i K_i)x(t) + A_{di}x(t-d) + D_i f_i(x(t), t). \quad (33)$$

Consider the following Lyapunov functional candidate:

$$V_i(t) = x^T(t)P_i x(t).$$

Along the trajectory of the system (33), the time derivative of $V_i(t)$ is given by

$$\begin{aligned} \dot{V}_i(t) &= 2\dot{x}^T(t)P_i x(t) \\ &= x^T(t) \left[(A_i + B_i M_i K_i)^T P_i \right. \\ &\quad \left. + P_i (A_i + B_i M_i K_i) \right] x(t) \\ &\quad + x^T(t) P_i A_{di} x(t-d) + x^T(t-d) A_{di}^T P_i x(t) \\ &\quad + 2x^T(t) P_i D_i f(x(t), t). \end{aligned}$$

From Lemma 2 and (5), we have

$$\begin{aligned} \dot{V}_i(t) &\leq x^T(t) \left[(A_i + B_i M_i K_i)^T P_i + P_i (A_i + B_i M_i K_i) \right. \\ &\quad \left. + P_i A_{di} P_i^{-1} A_{di}^T P_i + P_i D_i D_i^T P_i \right] x(t) \\ &\quad + x^T(t-d) P_i x(t-d) \\ &\quad + f_i^T(x(t), t) f_i(x(t), t) \\ &\leq x^T(t) \left[(A_i + B_i M_i K_i)^T P_i \right. \\ &\quad \left. + P_i (A_i + B_i M_i K_i) + U_i^T U_i \right. \\ &\quad \left. + P_i D_i D_i^T P_i + P_i A_{di} P_i^{-1} A_{di}^T P_i \right] x(t) \\ &\quad + x^T(t-d) P_i x(t-d). \end{aligned}$$

By Lemma 5, (18) is equivalent to

$$(A_i X_i + B_i M_i Y_i)^T + A_i X_i + B_i M_i Y_i + (1 + \alpha) X_i + A_{di} X_i A_{di}^T + D_i D_i^T + X_i U_i^T U_i X_i < 0. \quad (34)$$

Substituting $X_i = P_i^{-1}$, $K_i = Y_i X_i^{-1}$ to (34) and using P_i , pre- and postmultiply the left term of (34) to obtain

$$(A_i + B_i M_i K_i)^T P_i + P_i (A_i + B_i M_i K_i) + U_i^T U_i + P_i D_i D_i^T P_i + P_i A_{di} P_i^{-1} A_{di}^T P_i + (1 + \alpha) P_i < 0. \quad (35)$$

Then, by (35), we have

$$\begin{aligned} \dot{V}_i(t) &\leq -x^T(t) (1 + \alpha) P_i x(t) + x^T(t-d) P_i x(t-d) \\ &\leq -(1 + \alpha) V_i(t) + \sup_{-d \leq \theta_1 \leq 0} V_i(t + \theta_1). \end{aligned} \quad (36)$$

By Lemma 1, we have

$$\begin{aligned} V_i(t) &\leq \sup_{-d \leq \theta_1 \leq 0} V_i(t_{k-1} + \Delta_{k-1} + \theta_1) e^{-\mu_1(t-t_{k-1}-\Delta_{k-1})}, \end{aligned} \quad (37)$$

where $\mu_1 > 0$, and satisfies $\mu_1 + e^{\mu_1 d} = 1 + \alpha$.

Let

$$\kappa_1 = \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}.$$

We have

$$\begin{aligned} \|x(t)\| &\leq \kappa_1^{\frac{1}{2}} \sup_{-d \leq \theta_1 \leq 0} \|x(t_{k-1} + \Delta_{k-1} + \theta_1)\| \\ &\quad \cdot e^{-\frac{1}{2} \mu_1(t-t_{k-1}-\Delta_{k-1})}. \end{aligned} \quad (38)$$

When $t \in [t_k, t_k + \Delta_k)$, the closed-loop system (16)–(17) can be written as

$$\dot{x}(t) = (A_j + B_j M_j K_j)x(t) + A_{dj}x(t-d) + D_j f_j(x(t), t). \quad (39)$$

Consider the following Lyapunov functional candidate:

$$V_{ij}(t) = x^T(t) P_{ij} x(t).$$

Repeating the above proof line, from (19) we have

$$V_{ij}(t) \leq \sup_{-d \leq \theta_2 \leq 0} V_{ij}(t_k + \theta_2) e^{-\mu_2(t-t_k)}, \quad (40)$$

where $\mu_2 > 0$, and satisfies $\mu_2 + e^{\mu_2 d} = 1 + \eta$.

Let

$$\kappa_2 = \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(P_{ij})}.$$

We have

$$\|x(t)\| \leq \kappa_2^{\frac{1}{2}} \sup_{-d \leq \theta_2 \leq 0} \|x(t_k + \theta_2)\| e^{-\frac{1}{2} \mu_2(t-t_k)}. \quad (41)$$

Choosing $\mu = \min\{\mu_1, \mu_2\}$, we have

$$\begin{aligned} V_{\sigma(t_{k-1})}(t) &\leq \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_{k-1})}(t_{k-1} + \Delta_{k-1} + \theta_1) \\ &\quad \cdot e^{-\mu(t-t_{k-1}-\Delta_{k-1})}, \quad t \geq t_{k-1} + \Delta_{k-1}, \end{aligned} \quad (42)$$

$$\begin{aligned} V_{\sigma(t_{k-1})\sigma(t_k)}(t) &\leq \sup_{-d \leq \theta_2 \leq 0} V_{\sigma(t_{k-1})\sigma(t_k)}(t_k + \theta_2) \\ &\quad \cdot e^{-\mu(t-t_k)}, \quad t \geq t_k. \end{aligned} \quad (43)$$

Let

$$\rho_1 = \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_{ij})} \right\}.$$

Then we have

$$V_{\sigma(t_k)}(t) \leq \rho_1 V_{\sigma(t_{k-1})\sigma(t_k)}(t). \quad (44)$$

Let

$$\rho_2 = \max_{\substack{i,j \in \underline{N} \\ i \neq j}} \left\{ \frac{\lambda_{\max}(P_{ij})}{\lambda_{\min}(P_i)} \right\},$$

for $\theta_2 \in [-d, 0]$. We have

$$\begin{aligned} V_{\sigma(t_{k-1})\sigma(t_k)}(t_k + \theta_2) &\leq \rho_2 V_{\sigma(t_{k-1})}(t_k + \theta_2) \\ &\leq \rho_2 e^{\mu d} \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_{k-1})}(t_{k-1} + \Delta_{k-1} + \theta_1) \\ &\quad \cdot e^{-\mu(t_k-t_{k-1})} e^{\mu \Delta_{k-1}}. \end{aligned} \quad (45)$$

Notice that $-d \leq \Delta_k + \theta_1 \leq d$, so we can obtain

$$\begin{aligned} V_{\sigma(t_{k-1})}(t_{k-1} + \Delta_{k-1} + \theta_1) &\leq \rho_1 \rho_2 e^{2\mu d} \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_{k-2})}(t_{k-2} + \Delta_{k-2} + \theta_1) \\ &\quad \cdot e^{-\mu[(t_{k-1} + \Delta_{k-1}) - (t_{k-2} + \Delta_{k-2})]} \\ &\leq (\rho_1 \rho_2 e^{2\mu d})^{k-1} e^{-\mu(t_{k-1}-t_0)} e^{-\mu(\Delta_{k-1}-\Delta_0)} \\ &\quad \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_0)}(t_0 + \Delta_0 + \theta_1), \end{aligned} \quad (46)$$

which leads to

$$\begin{aligned}
 &V_{\sigma(t_{k-1})}(t) \\
 &\leq (\rho_1 \rho_2 e^{2\mu d})^{k-1} e^{-\mu(t_{k-1}-t_0)} e^{-\mu(t-t_{k-1}-\Delta_{k-1})} \\
 &\quad \cdot e^{-\mu(\Delta_{k-1}-\Delta_0)} \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_0)}(t_0 + \Delta_0 + \theta_1).
 \end{aligned} \tag{47}$$

From $t_{k+1} - t_k \geq T$, we have

$$t - t_0 - \Delta_0 \geq (k - 1)T - d. \tag{48}$$

Let

$$\begin{aligned}
 &T > 2d + \frac{\ln \rho_1 \rho_2}{\mu}, \\
 &\nu = -\frac{1}{2} \left(\frac{\ln \rho_1 \rho_2 + 2d\mu}{T} - \mu \right) > 0.
 \end{aligned}$$

Then

$$\begin{aligned}
 V_{\sigma(t_{k-1})}(t) &\leq \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_0)}(t_0 + \Delta_0 + \theta_1) \\
 &\quad \cdot e^{(\frac{\ln \rho_1 \rho_2 + 2\mu d}{T} - \mu)(t - t_0 - \Delta_0)}.
 \end{aligned} \tag{49}$$

Similarly, we have

$$\begin{aligned}
 &V_{\sigma(t_{k-1})\sigma(t_k)}(t) \\
 &\leq \rho_1^{-1} (\rho_1 \rho_2 e^{2\mu d})^{\frac{k}{T}} \sup_{-d \leq \theta_1 \leq 0} V_{\sigma(t_0)}(t_0 + \Delta_0 + \theta_1) \\
 &\quad \cdot e^{(\frac{\ln \rho_1 \rho_2 + 2\mu d}{T} - \mu)(t - t_0 - \Delta_0)}
 \end{aligned} \tag{50}$$

The proof is completed.

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