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SINGULAR FRACTIONAL LINEAR SYSTEMS AND ELECTRICAL CIRCUITS

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A new class of singular fractional linear systems and electrical circuits is introduced. Using the Caputo definition of the fractional derivative, the Weierstrass regular pencil decomposition and the Laplace transformation, the solution to the state equation of singular fractional linear systems is derived. It is shown that every electrical circuit is a singular fractional system if it contains at least one mesh consisting of branches only with an ideal supercapacitor and voltage sources or at least one node with branches with supercoils.

Keywords: singular, fractional system, linear circuit, regular pencil, supercapacitor, supercoil.

1. Introduction

Singular (descriptor) linear systems were addressed in many papers and books (Dodig and Stosic, 2009; Dai, 1989; Fahmy and O'Reill, 1989; Kaczorek, 1992; 2004; 2007a; 2007b; Kucera and Zagalak, 1988; Van Dooren, 1979). The eigenvalues and invariants assignment by state and output feedbacks were investigated by Dodig and Stosic (2009), Dai (1989), Fahmy and O'Reill (1989), or Kaczorek (1992; 2004), and the realization problem for singular positive continuous-time systems with delays by Kaczorek (2007b). The computation of Kronecker's canonical form of a singular pencil was analyzed by Van Dooren (1979).

Fractional positive continuous-time linear systems were addressed by Kaczorek (2008) along with positive linear systems with different fractional orders (Kaczorek, 2010). An analysis of fractional linear electrical circuits was presented in another work of Kaczorek (2011) and some selected problems in theory of fractional linear systems in a monograph by the same author (Kaczorek, 2009).

In this paper a new class of singular fractional linear systems and electrical circuits will be introduced and their solution of state equations will be derived. The paper is organized as follows. In Section 2 the Caputo definition of the fractional derivative and the solution to the state equation of the fractional linear system are recalled. The solution of the state equation of a singular fractional linear system is derived in Section 3 using the Weirstrass pencil decomposition and the Laplace transform. Singular fractional linear electrical circuits are introduced in Section 4. Concluding remarks are given in Section 5.

To the best of the author's knowledge, singular fractional linear systems and electrical circuits have not been considered yet.

The following notation will be used in the paper. The set of $n \times m$ real matrices will be denoted by $\mathbb{R}^{n \times m}$ and $\mathbb{R}^n := \mathbb{R}^{n \times 1}$. The set of $m \times n$ real matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{m \times n}$ and $\mathbb{R}_+^n := \mathbb{R}_+^{n \times 1}$. The set of nonnegative integers will be denoted by $\mathbb{Z}_+$ and the $n \times n$ identity matrix by I_n.

2. Preliminaries

The following Caputo definition of the fractional derivative will be used (Podlubny, 1999; Kaczorek, 2009):

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} \, \mathrm{d}\tau, \quad (1)$$
$$n - 1 < \alpha \le n \in \mathbb{N} = \{1, 2, \dots\},$$

where $\alpha \in \mathbb{R}$ is the order of the fractional derivative,

$$f^{(n)}(\tau) = \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n},$$

and

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, \mathrm{d}t$$

is the gamma function.

Consider the continuous-time fractional linear system described by the state equation

$$\frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \le 1, \qquad (2)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and input vectors, respectively, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Theorem 1. *The solution of Eqn. (2) is given by*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t - \tau)Bu(t)\,d\tau, \quad x(0) = x_0, \qquad (3)$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \qquad (4)$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}. \qquad (5)$$

The proof was given by Kaczorek (2008; 2009).

Remark 1. From (4) and (5) for $\alpha = 1$ we have

$$\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(k+1)} = e^{At}.$$

3. Singular fractional linear systems

Consider a singular fractional linear system described by the state equations

$$E\frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) + Bu(t), \qquad (6a)$$

$$y(t) = Cx(t) + Du(t), \qquad (6b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are respectively the state, input and output vectors, and $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

The initial condition for (6a) is given by

$$x(0) = x_0. \qquad (6c)$$

It is assumed that the pencil of the pair (E, A) is regular, i.e.,

$$\det[Es - A] \ne 0, \qquad (7)$$

for some $s \in \mathbb{C}$ (the field of complex numbers). It is well known (Gantmacher, 1960; Kaczorek, 2007a, p. 92) that, if the pencil is regular, then there exists a pair of nonsingular matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$P[Es - A]Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} s - \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \qquad (8)$$

where n_1 is equal to the degree of the polynomial $\det[Es - A]$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix with the index μ (i.e., $N^\mu = 0$ and $N^{\mu-1} \ne 0$) and $n_1 + n_2 = n$.

Applying the Laplace transform $(\mathcal{L})$ to Eqn. (6a) with zero initial conditions $x_0 = 0$, we obtain

$$[Es^\alpha - A]X(s) = BU(s), \qquad (9)$$

where

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}\,dt$$

and $U(s) = \mathcal{L}[u(t)]$. By the assumption (7), the pencil $[Es^\alpha - A]$ is regular and we may apply the decomposition (8) to Eqn. (6a).

Premultiplying (6a) by the matrix $P \in \mathbb{R}^{n \times n}$ and introducing the new state vector

$$\bar{x}(t) = Q^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad (10)$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, we obtain

$$\frac{d^\alpha}{dt^\alpha} x_1(t) = A_1 x_1(t) + B_1 u(t), \qquad (11a)$$

$$N\frac{d^\alpha}{dt^\alpha} x_2(t) = x_2(t) + B_2 u(t), \qquad (11b)$$

where

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1 \times m}, B_2 \in \mathbb{R}^{n_2 \times m}. \quad (11c)$$

Using (3) we obtain the solution to Eqn. (11a) in the form

$$x_1(t) = \Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t - \tau)B_1 u(t)\,d\tau, \quad (12a)$$

where

$$\Phi_{10}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \qquad (12b)$$

$$\Phi_{11}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}, \qquad (12c)$$

and $x_{10} \in \mathbb{R}^{n_1}$ is the initial condition for (11a) defined by

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = Q^{-1}x_0, \quad x_0 = x(0). \qquad (12d)$$

To find the solution of Eqn. (11b), we apply the Laplace transform and obtain

$$Ns^\alpha X_2(s) - Ns^{\alpha-1}x_{20} = X_2(s) + B_2 U(s) \quad (13a)$$

since (Dai, 1989; Kaczorek, 2008), for $0 < \alpha < 1$,

$$\mathcal{L}\left[\frac{d^\alpha}{dt^\alpha} x_2(t)\right] = s^\alpha X_2(s) - s^{\alpha-1}x_{20}, \qquad (13b)$$

where $X_2(s) = \mathcal{L}[x_2(t)]$. From (13) we have

$$X_2(s) = [Ns^\alpha - I_{n_2}]^{-1}(B_2 U(s) + Ns^{\alpha-1}x_{20}). \quad (14)$$

It is easy to check that

$$[Ns^\alpha - I_{n_2}]^{-1} = -\sum_{i=0}^{\mu-1} N^i s^{i\alpha} \quad (15)$$

since

$$[Ns^\alpha - I_{n_2}]\left(-\sum_{i=0}^{\mu-1} N^i s^{i\alpha}\right) = I_{n_2} \quad (16)$$

and $N^i = 0$ for $i = \mu, \mu+1, \ldots$.

Substitution of (15) into (14) yields

$$X_2(s) = -B_2 U(s) - \frac{Nx_{20}}{s^{1-\alpha}}$$
$$- \sum_{i=1}^{\mu-1}\left[N^i B_2 s^{i\alpha} U(s) + N^{i+1} s^{(i+1)\alpha-1} x_{20}\right]. \quad (17)$$

Applying the inverse Laplace transform ($\mathcal{L}^{-1}$) to (17) and then the convolution theorem we obtain, for $1 - \alpha > 0$,

$$x_2(t) = \mathrm{L}^{-1}[X_2(s)]$$
$$= -B_2 u(t) - Nx_{20}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$
$$- \sum_{i=1}^{\mu-1}\left[N^i B_2 \frac{\mathrm{d}^{i\alpha}}{\mathrm{d}t^{i\alpha}}u(t) + N^{i+1}\frac{\mathrm{d}^{(i+1)\alpha-1}}{\mathrm{d}t^{(i+1)\alpha-1}}x_{20}\right] \quad (18)$$

since

$$\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(1+\alpha)}$$

for $\alpha + 1 > 0$.

Therefore, the following theorem has been proved.

Theorem 2. *The solution to Eqn. (6a) with the initial condition (6c) has the form*

$$x(t) = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (19)$$

where $x_1(t)$ and $x_2(t)$ are given by (12) and (18), respectively.

Knowing the solution (19), we can find the output $y(t)$ of the system using the formula

$$y(t) = CQ\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t). \quad (20)$$

4. Singular fractional electrical circuits

Let the current $i_C(t)$ in the supercapacitor with the capacity C be the α-th order derivative of its charge $q(t)$ (Kaczorek, 2011),

$$i_C(t) = \frac{\mathrm{d}^\alpha q(t)}{\mathrm{d}t^\alpha}. \quad (21)$$

Taking into account that $q(t) = Cu_C(t)$, we obtain

$$i_C(t) = C\frac{\mathrm{d}^\alpha u_C(t)}{\mathrm{d}t^\alpha}, \quad (22)$$

where $u_C(t)$ is the capacitor voltage.

Similarly, let the voltage $u_L(t)$ of the supercoil (inductor) with the inductance L be the β-th order derivative of its magnetic flux $\psi(t)$,

$$u_L(t) = \frac{\mathrm{d}^\beta \Psi(t)}{\mathrm{d}t^\beta}. \quad (23)$$

Taking into account that $\psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{\mathrm{d}^\beta i_L(t)}{\mathrm{d}t^\beta}, \quad (24)$$

where $i_L(t)$ is the current of the supercoil.

Example 1. Consider the electrical circuit shown in Fig. 1 with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.
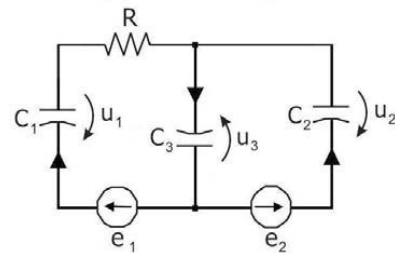

Fig. 1. Electrical circuit of Example 1.

Using Kirchhoff's laws, for the electrical circuit we can write the equations

$$e_1 = RC_1\frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + u_1 + u_3,$$
$$C_1\frac{\mathrm{d}^\alpha u_1}{\mathrm{d}t^\alpha} + C_2\frac{\mathrm{d}^\alpha u_2}{\mathrm{d}t^\alpha} - C_3\frac{\mathrm{d}^\alpha u_3}{\mathrm{d}t^\alpha} = 0, \quad (25)$$
$$e_2 = u_2 + u_3.$$

They (25) can be rewritten in the form

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (26)$$

In this case we have

$$
E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \tag{27}
$$

$$
B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Note that the matrix E is singular ($\det E = 0$) but the pencil

$$
\begin{aligned}
&\det[Es^\alpha - A] \\
&= \begin{vmatrix} RC_1 s^\alpha + 1 & 0 & 1 \\ C_1 s^\alpha & C_2 s^\alpha & -C_3 s^\alpha \\ 0 & 1 & 1 \end{vmatrix} \\
&= (RC_1 s^\alpha + 1)(C_2 + C_3)s^\alpha + C_1 s^\alpha
\end{aligned} \tag{28}
$$

is regular. Therefore, the electrical circuit is a singular fractional linear system. ♦

Remark 2. If the electrical circuit contains at least one mesh consisting of branches with only ideal supercapacitors and voltage sources, then its matrix E is singular since the row corresponding to this mesh is a zero row. This follows from the fact that the equation written with the use of Kirchhoff's voltage law is an algebraic one.

Example 2. Consider the electrical circuit shown in Fig. 2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 and source voltages e_1 and e_2.
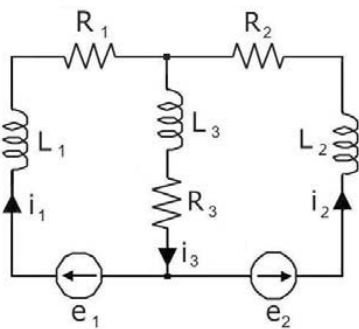

Fig. 2. Electrical circuit of Example 2.

Using Kirchhoff's laws we can write, the equations

$$
\begin{aligned}
e_1 &= R_1 i_1 + L_1 \frac{\mathrm{d}^\beta i_1}{\mathrm{d}t^\beta} + R_3 i_3 + L_3 \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta}, \\
e_2 &= R_2 i_2 + L_2 \frac{\mathrm{d}^\beta i_2}{\mathrm{d}t^\beta}, + R_3 i_3 + L_3 \frac{\mathrm{d}^\beta i_3}{\mathrm{d}t^\beta} \\
i_1 &+ i_2 - i_3 = 0.
\end{aligned} \tag{29}
$$

Equations (29) can be written in the form

$$
\begin{aligned}
&\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \\
&= \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.
\end{aligned} \tag{30}
$$

In this case we have

$$
E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \tag{31}
$$

$$
B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
$$

Note that the matrix E is singular but the pencil

$$
\begin{aligned}
&\det[Es^\beta - A] \\
&= \begin{vmatrix} L_1 s^\beta + R_1 & 0 & L_3 s^\beta + R_3 \\ 0 & L_2 s^\beta + R_2 & L_3 s^\beta + R_3 \\ -1 & -1 & 1 \end{vmatrix} \\
&= [L_1(L_2 + L_3) + L_2 L_3]s^{2\beta} \\
&\quad + [(L_2 + L_3)R_1 + (L_1 + L_3)R_2 + (L_1 + L_2)R_3]s^\beta \\
&\quad + R_1(R_2 + R_3) + R_2 R_3
\end{aligned} \tag{32}
$$

is regular. Therefore, the electrical circuit is a singular fractional linear system. ♦

Remark 3. If the electrical circuit contains at least one node with branches with supercoils, then its matrix E is singular since it has at least one zero row. This follows from the fact that the equation written using Kirchhoff's current law for this node is an algebraic one.

In the general case we have the following theorem.

Theorem 3. *Every electrical circuit is a singular fractional system if it contains at least one mesh consisting of branches with only ideal supercapacitors and voltage sources or at least one node with branches with supercoils.*

Proof. By Remark 1 the matrix E of the system is singular if the electrical circuit contains at least one mesh consisting of branches with only ideal supercapacitors and voltage sources. Similarly, by Remark 2 the matrix E is singular if the electrical circuit contains at least one node with branches with supercoils. ∎

Using the solution (19) of Eqn. (6a) we may find the voltages on the supercapacitors and currents in the super-coils in transient states of singular fractional linear electrical circuits. Knowing the voltages and currents and using (20), we may also find any currents and voltages in singular fractional linear electrical circuits.

Example 3. (Continuation of Example 1) Using one of the well-known methods (Van Dooren, 1979; Dai, 1989; Kaczorek, 2007a) for the pencil (28), we can find the matrices

$$P = \begin{bmatrix} \frac{1}{RC_1} & 0 & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{1}{C_2+C_3} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & 0 & -1 \end{bmatrix}, \tag{33}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{C_3}{C_2+C_3} \\ 0 & -1 & \frac{C_2}{C_2+C_3} \end{bmatrix},$$

which transform it to the canonical form (8) with

$$A_1 = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{R(C_2+C_3)} & -\frac{1}{R(C_2+C_3)} \end{bmatrix},$$

$$N = [0], \quad n_1 = 2, \quad n_2 = 1. \tag{34}$$

Using the matrix B given by (27), (33) and (11c) we obtain

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB$$

$$= \begin{bmatrix} \frac{1}{RC_1} & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{C_2}{R(C_2+C_3)} \\ 0 & -1 \end{bmatrix}. \tag{35}$$

From (12) we have

$$x_1(t) = \Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1 u(t)\,\mathrm{d}\tau \tag{36}$$

for any given initial condition $x_{10} \in \mathbb{R}^{n_1}$ and input $u(t)$, where

$$\Phi_{10}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)},$$

$$\Phi_{11}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \quad 0 < \alpha < 1.$$

In this case, using (18) we obtain

$$x_2(t) = -B_2 u(t) \tag{37}$$

since $N = [0]$.

In much the same way we may find currents in the su-percoils of the singular fractional electrical circuit shown in Fig. 2.

5. Concluding remarks

Singular fractional linear systems and electrical circuits have been introduced. Using the Caputo definition of the fractional derivative, the Weierstrass regular pencil decomposition and the Laplace transform, the solution to the state equation of singular fractional linear system was derived (Theorem 2). Singular fractional linear electrical circuits were analyzed. It was shown that every electrical circuit is a singular fractional system if it contains at least one mesh consisting of branches with only ideal supercapacitors and voltage sources or at least one node with branches with supercoils (Theorem 3). The discussion was illustrated by singular linear electrical circuits. It can be extended to singular fractional linear systems with singular pencils.

An open problem is extension of the results to positive singular fractional linear systems and singular positive linear systems with different fractional orders. Linear systems with different fractional orders are described by (Kaczorek, 2010)

$$\begin{bmatrix} \frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} \\ \frac{\mathrm{d}^\beta x_2}{\mathrm{d}t^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,$$

$$p-1 < \alpha < p, \quad q-1 < \beta < q, \quad p,q \in \mathbb{N}, \tag{38}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ are the state vectors and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$, $i, j = 1,2$, and $u \in \mathbb{R}^m$ is the input vector. The initial conditions for (38) have the form $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$.

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