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# REGIONAL CONTROL PROBLEM FOR DISTRIBUTED BILINEAR SYSTEMS: APPROACH AND SIMULATIONS

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This paper investigates the regional control problem for infinite dimensional bilinear systems. We develop an approach that characterizes the optimal control and leads to a numerical algorithm. The obtained results are successfully illustrated by simulations.

**Keywords:** distributed systems, bilinear systems, regional controllability, regional optimal control problem.

## 1. Introduction

The aim of several control problems is to drive a dynamical system from an initial state to a desired one in a finite time. Let us consider a distributed bilinear systems evolving on $\Omega \subset \mathbb{R}^n$ and described by the equation

$$
\begin{cases}
\dot{z}(t) = Az(t) + u(t)Bz(t), & t \in [0, T], \\
z(0) = z_0 \neq 0,
\end{cases}
\tag{1}
$$

where $A$ is the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$ on the state space $Z =: L^2(\Omega)$ endowed with its natural inner product $\langle \cdot, \cdot \rangle$, and the corresponding norm $\| \cdot \|$, $B : Z \to Z$ is a linear bounded operator, while $u \in L^2[0, T]$ is a control. The main result on the controllability of the system (1) is due to the pioneering work by Ball *et al.* (1982), which shows that, under the above-mentioned conditions, a mild solution $z_u$ of (1) associated with the control $u$ exists and the set of reachable states from an initial state $z_0$ is of dense complement in the state space. This makes exact controllability difficult to be achieved.

Most results are established for particular bilinear systems (Ball *et al.*, 1982; Joshi, 2005; Lenhart and Liang, 2000; Khapalov, 2002a; 2002b). Later the concept of regional controllability for linear distributed systems has been introduced and developed by El Jai and Zerrik and

concerns the transfer of such a system to a desired state only on a region of the system spacial domain. The system (1) is said to be exactly (respectively, approximately) controllable in $\omega \subset \Omega$ if for all $z_d \in L^2(\omega)$ there exists a control $u \in L^2[0, T]$ such that $\chi_\omega z_u(T) = z_d$ (respectively, $\|\chi_\omega z_u(T) - z_d\|_{L^2(\omega)} \leq \varepsilon, \varepsilon > 0$), where $z_d$ is a desired state in the space $L^2(\omega)$, $\chi_\omega : Z \longrightarrow L^2(\omega)$ is the restriction operator to $\omega$. Many results for linear and semi linear systems have been developed (see El Jai *et al.*, 1995; Zerrik and Kamal, 2007; Zerrik *et al.*, 2007).

This concept finds its applications in many real world problems. For example, the physical problem which concerns a tunnel furnace where one has to maintain a prescribed temperature only in a subregion of the furnace. Also there exist systems which are controllable on some subregion $\omega \subset \Omega$ but not controllable in the whole domain $\Omega$ and that controlling regionally a system is cheaper than controlling it in the whole domain (see El Jai *et al.*, 1995). In this paper we discuss an extension of previous works (El Jai *et al.*, 1995; Zerrik and Kamal, 2007; Zerrik *et al.*, 2007) on regional controllability for linear and semi linear systems to a bilinear one. More precisely, for the system (1) defined on a spatial domain $\Omega$, a nonempty subset $\omega \subset \Omega$, with a positive Lebesgue measure and a desired state $z_d$ in $L^2(\omega)$, the problem of regional controllability for (1) consists in finding a control function with minimum energy in an appropriate control space that

steers (1) from $z_0$ to a final state close to $z_d$ on $\omega$ at time $T$. This problem may be stated as follows:

$$
\begin{cases}
\text{Find } u \in L^2[0,T] \text{ which minimizes } \|u\|^2_{L^2[0,T]}, \\
u \in U_{\text{ad}}(\omega),
\end{cases}
$$

$$(2)$$

while

$$
\begin{aligned}
& U_{\text{ad}}(\omega) \\
& = \{u \in L^2[0,T] : \|\chi_{\omega} z_u(T) - z_d\|_{L^2(\omega)} \text{ is minimum}\}.
\end{aligned}
$$

We discuss the cases of $U_{\text{ad}}(\omega) \neq \emptyset$ and $U_{\text{ad}}(\omega) = \emptyset$. To characterize the optimal solution of (2), we propose an approach based on a quadratic cost control problem, which involves the minimization of the control norm and the final state error. This is the aim of this paper, which is organized as follows. In Section 2 we consider the quadratic cost control problem associated with (2). In Section 3, we give a characterization of a control solution of (2) and we show that, under supplementary conditions, the uniqueness may be ensured. In the last section, we develop a numerical approach and give illustrations with numerical examples and simulations.

## 2. Regional quadratic control problem

Given $T > 0$, let us associate with (2) the problem

$$
\min_{u \in L^2[0,T]} J_{\epsilon}(u)
$$

$$(3)$$

with

$$
\begin{aligned}
J_{\epsilon}(u) = & \|\chi_{\omega} z_u(T) - z_d\|^2_{L^2(\omega)} \\
& + \epsilon \int_0^T u^2(t)\,\mathrm{d}t, \quad \epsilon > 0.
\end{aligned}
$$

$$(4)$$

**Proposition 1.**

*1. For $u \in L^2[0,T]$ and $h \in L^2[0,T]$, $\forall t \in [0,T]$, we have $\|z_{u+h}(t) - z_u(t)\| = o(\|h\|)$ as $h \to 0$.*

*2. There exists $u^* \in L^2[0,T]$ such that*

$$
J_{\varepsilon}(u^*) = J^* = \min_{v \in L^2[0,T]} J_{\varepsilon}(v).
$$

*Let*

$$
y(t) = \int_0^t U(t,s)h(s)Bz_u(s)\,\mathrm{d}s.
$$

*Then*

$$
\|z_{u+h}(t) - z_u(t) - y(t)\| = o(\|h\|)
$$

*as $h \to 0$, where $(U(t,s))_{t \geq s}$ is the evolution operator generated by $A + uB$.*

*Proof.*
1. We have

$$
\begin{aligned}
& z_{u+h}(t) - z_u(t) \\
& = \int_0^t S(t-s)u(s)B(z_{u+h}(s) - z_u(s))\,\mathrm{d}s \\
& \quad + \int_0^t S(t-s)h(s)Bz_{u+h}(s)\,\mathrm{d}s.
\end{aligned}
$$

Using the boundedness of the semigroup $(S(t))_{t \geq 0}$ on the entire finite interval of $[0,T]$, i.e., the fact that there is an $M > 0$ such that $\|S(t)\| \leq M$, $\forall t \in [0,T]$, we have

$$
\begin{aligned}
& \|z_{u+h}(t) - z_u(t)\| \\
& \leq M\|B\|\Big(\int_0^t \big[|u(s)|\|z_{u+h}(s) - z_u(s)\| \\
& \quad + |h(s)|\|z_{u+h}(s)\|\big]\,\mathrm{d}s\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\|z_{u+h}(t)\| \leq M\Big(&\|z_0\| \\
& + \|B\| \int_0^t |u(s) + h(s)|\|z_{u+h}(s)\|\,\mathrm{d}s\Big).
\end{aligned}
$$

Using the Gronwall inequality twice, we obtain $\|z_{u+h}(t)\| \leq k_1$, and

$$
\begin{aligned}
& \|z_{u+h}(t) - z_u(t)\| \\
& \leq M\|B\|\Big(\int_0^t |u(s)|\|z_{u+h}(s) - z_u(s)\|\,\mathrm{d}s \\
& \quad + k_1 \int_0^t |h(s)|\,\mathrm{d}s\Big),
\end{aligned}
$$

and, again by the Gronwall inequality, we obtain $\|z_{u+h}(t) - z_u(t)\| = o(\|h\|)$ as $h \to 0$.

2. The set $E = \{J_{\varepsilon}(u) \mid u \in L^2[0,T]\}$ is nonempty and bounded from below, so the lower bound $J^*$ exists. Let the sequence $(u_n)$ in $L^2[0,T]$ be such that

$$
\lim_{n \to +\infty} J_{\varepsilon}(u_n) = J^*.
$$

We have

$$
\varepsilon \int_0^T u_n^2(t)\,\mathrm{d}t \leq J_{\varepsilon}(u_n).
$$

Hence $(u_n)_{n \geq 0}$ is bounded. Therefore, we can extract a subsequence denoted by $(u_{n_k})$ which converges weakly to $u^*$ in $L^2[0,T]$. This implies that $z_{u_{n_k}}$ converges to $z_{u^*}$ strongly in $\mathcal{C}(0,T;Z)$ (see Ball *et al.*, 1982). Hence

$$
J_{\varepsilon}(u^*) \leq \liminf_{n \to \infty} J_{\varepsilon}(u_n) = J^* \leq J_{\varepsilon}(u^*).
$$

3. Since $u \in L^2[0,T]$ and $B$ is a bounded linear operator on $Z$, the operator $A + uB \in L^1[0,T; \ D(A)]$. Then

$A + u(t)B$ generates an evolution operator $(U(t,s))_{t \geq s}$ (cf. Pazy, 1983, Chapter 5, Remark 3.2). Thus

$$y(t) = \int_0^t U(t,s)h(s)Bz_u(s)\,\mathrm{d}s$$

is well defined.

Let $Y(t) = z_{u+h}(t) - z_u(t) - y(t)$. We can write

$$Y(t) = \int_0^t S(t-s)u(s)BY(s)\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)h(s)B(z_{u+h}(s) - z_u(s))\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)h(s)Bz_u(s)\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)u(s)By(s)\,\mathrm{d}s - y(t).$$

Let

$$K(t) = \int_0^t S(t-s)h(s)Bz_u(s)\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)u(s)By(s)\,\mathrm{d}s - y(t).$$

Then, for $z_0 \in D(A)$, we have

$$\dot{K}(t) = A\int_0^t S(t-s)h(s)Bz_u(s)\,\mathrm{d}s$$
$$+ h(t)Bz_u(t) + A\int_0^t S(t-s)u(s)By(s)\,\mathrm{d}s$$
$$+ u(t)By(t) - \dot{y}(t).$$

Since $\dot{y}(t) = (A + u(t)B)y(t) + h(t)Bz_u(t)$ and $y(0) = 0$, we get

$$y(t) = \int_0^t S(t-s)h(s)Bz_u(s)\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)u(s)By(s)\,\mathrm{d}s,$$

which shows that $\dot{K}(t) = 0$, and since $K(0) = 0$, it follows that $K(t) = 0$, $\forall t \in [0,T]$.

Then we have

$$Y(t) = \int_0^t S(t-s)u(s)BY(s)\,\mathrm{d}s$$
$$+ \int_0^t S(t-s)h(s)B(z_{u+h}(s) - z_u(s))\,\mathrm{d}s,$$

and

$$\|Y(t)\| \leq M\|B\| \left( \int_0^t |u(s)|\|Y(s)\|\,\mathrm{d}s \right.$$
$$\left. + \int_0^t |h(s)|\|z_{u+h}(s) - z_u(s)\|\,\mathrm{d}s \right).$$

By Property 1, we have

$$M\|B\| \int_0^t |h(t)|\|z_{u+h}(s) - z_u(s)\|\,\mathrm{d}s$$
$$\leq k_1\|h\|^2, \quad k_1 \in \mathbb{R}.$$

By the Gronwall inequality, we obtain

$$\|Y(t)\| \leq k_2\|h\|^2, \quad k_2 \in \mathbb{R},$$

that is,

$$\|Y(t)\| = o(\|h\|),$$

and by the density of $D(A)$ in $Z$ we have the above inequality in $Z$. ∎

Now, the solution to the problem (3) is characterized by the following result.

**Theorem 1.** *A control which minimizes the problem (3) is given by*

$$u(t) = -\frac{1}{\epsilon}\langle Bz(t), P(t)z(t) - U^*(T,t)\chi_\omega^* z_d\rangle, \quad (5)$$

*where $P$ is the selfadjoint and nonnegative operator solution of the following equation:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle P(t)y, z\rangle + \langle P(t)y, (A + u(t)B)z\rangle \\ + \langle (A + u(t)B)y, P(t)z\rangle = 0, \quad \forall y, z \in D(A), \\ P(T) = \chi_\omega^* \chi_\omega. \end{cases}$$
$$(6)$$

*Here $U^*(t,s)$ is the adjoint operator of $U(t,s)$ and $\chi_\omega^*$ is the adjoint operator of $\chi_\omega$.*

*The minimum is given by*

$$J_\varepsilon(u) = \langle P(0)z_0, z_0\rangle + 2\langle \chi_\omega U(T,0)z_0, z_d\rangle$$
$$+ \|z_d\|_{L^2(\omega)}^2 + \varepsilon \int_0^T u^2(t)\,\mathrm{d}t.$$

*Proof.* Using Property 3 of the previous proposition, we have
$$z_{u+h}(t) = z_u(t) + y(t) + o(h).$$
Then we obtain
$$\langle \chi_\omega z_{u+h}(t) - z_d, \chi_\omega z_{u+h}(t) - z_d\rangle$$
$$= \langle \chi_\omega z_u(t) - z_d, \chi_\omega z_u(t) - z_d\rangle$$
$$+ 2\langle \chi_\omega z_u(t) - z_d, \chi_\omega y(t)\rangle + o(\|h\|).$$

Hence

$$J_\varepsilon(u+h) - J_\varepsilon(u) = 2\langle \chi_\omega z_u(T) - z_d, \chi_\omega y(T)\rangle$$
$$+ 2\varepsilon \int_0^T u(t)h(t)\,\mathrm{d}t + o(\|h\|).$$

For $v \in L^2[0, \ T]$, define

$$\Lambda v(t) = \chi_\omega U(T,t)v(t)Bz_u(t).$$

Since

$$y(T) = \int_0^T U(T,s)h(s)Bz_u(s)\,\mathrm{d}s,$$

we obtain

$$J_\varepsilon(u+h) - J_\varepsilon(u)$$
$$= 2\int_0^T [\langle\chi_\omega z_u(T) - z_d, \Lambda h(t)\rangle\varepsilon u(t)h(t)]\,\mathrm{d}t$$
$$+ o(\|h\|).$$

Thus, the differential of $J_\varepsilon$ at $u$ is

$$\mathrm{d}J_\varepsilon(u)h = 2\int_0^T \Big[\langle\Lambda^*(\chi_\omega z_u(T) - z_d), h(t)\rangle$$
$$+ \varepsilon u(t)h(t)\Big]\,\mathrm{d}t.$$

Then $\mathrm{d}J_\varepsilon(u)h = 0, \forall h \in L^2[0,T]$ is equivalent to

$$u(t) = -\frac{1}{\varepsilon}\Lambda^*(\chi_\omega z_u(T) - z_d).$$

Also, we have

$$\Lambda^*v(t) = z_u^*(t)B^*U^*(T,t)\chi_\omega^*v(t),$$

which gives

$$u(t) = -\frac{1}{\varepsilon}z_u^*(t)B^*U^*(T,t)(\chi_\omega^*\chi_\omega z_u(T) - \chi_\omega^* z_d).$$

Let us consider the following nonnegative and self-adjoint operator:

$$P(t)z = U^*(T,t)\chi_\omega^*\chi_\omega U(T,t)z, \quad \forall z \in D(A).$$

Since $z_u(t) = U(t,0)z_0$, we have

$$P(t)z_u(t) = U^*(T,t)\chi_\omega^*\chi_\omega z_u(T),$$

and then we obtain (5).

Let us show that $P(t)$ satisfies Eqn. (6). We have

$$\frac{\partial U}{\partial s}(t,s)z = -U(t,s)(A + u(s)B)z, \quad \forall z \in D(A).$$

Then, $\forall y, z \in D(A)$ and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle U^*(T,t)\chi_\omega^*\chi_\omega U(T,t)y, z\rangle$$
$$= -\langle\chi_\omega U(T,t)(A + u(t)B)y, \chi_\omega U(T,t)z\rangle$$
$$- \langle\chi_\omega U(T,t)y, \chi_\omega U(T,t)(A + u(t)B)z\rangle,$$

which shows the right part of (6). Now, using

$$z_u(t) = U(t,0)z_0,$$

$$\langle\chi_\omega z_u(T), \chi_\omega z_u(T)\rangle = \langle P(0)z_0, z_0\rangle,$$

we have

$$J_\varepsilon(u) = \langle P(0)z_0, z_0\rangle + 2\langle\chi_\omega U(T,0)z_0, z_d\rangle$$
$$+ \|z_d\|_{L^2(\omega)}^2 + \varepsilon\int_0^T u^2(t)\,\mathrm{d}t.$$

$\blacksquare$

**Remark 1.** Equation (6) has a unique solution (cf. El Alami, 1988).

If $u_\epsilon$ denotes the solution of (3) and $z_\epsilon$ the associated state of (1), the following result will be useful for the sequel of the paper.

**Proposition 2.**

1. *The sequence* $(J_\epsilon(u_\epsilon))_{\epsilon>0}$ *is decreasing as* $\epsilon \to 0$.

2. *The sequence*

$$\left(\int_0^T u_\epsilon^2(t)\,\mathrm{d}t\right)_{\epsilon>0}$$

*is increasing as* $\epsilon \to 0$.

3. *The sequence* $\left(\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2\right)_{\epsilon>0}$ *is decreasing as* $\epsilon \to 0$, *and* $\forall \epsilon > 0$

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)} \leq \|\chi_\omega S(T)z_0 - z_d\|_{L^2(\omega)}.$$

*In particular, there exists a subsequence of* $(\chi_\omega z_\epsilon(T) - z_d)_{\epsilon>0}$ *which converges weakly in* $L^2(\omega)$.

*Proof.* Let $0 < \epsilon_1 < \epsilon_2$. Using consecutively the optimality of $u_{\epsilon_1}$ for $J_{\epsilon_1}$ and the optimality of $u_{\epsilon_2}$ for $J_{\epsilon_2}$, we have

$$J_{\epsilon_1}(u_{\epsilon_1}) = \|\chi_\omega z_{\epsilon_1}(T) - z_d\|_{L^2(\omega)}^2 + \epsilon_1\int_0^T u_{\epsilon_1}^2(t)\,\mathrm{d}t$$
$$\leq \|\chi_\omega z_{\epsilon_2}(T) - z_d\|_{L^2(\omega)}^2 + \epsilon_1\int_0^T u_{\epsilon_2}^2(t)\,\mathrm{d}t$$
$$\leq \|\chi_\omega z_{\epsilon_2}(T) - z_d\|_{L^2(\omega)}^2 + \epsilon_2\int_0^T u_{\epsilon_2}^2(t)\,\mathrm{d}t$$
$$\leq \|\chi_\omega z_{\epsilon_1}(T) - z_d\|_{L^2(\omega)}^2 + \epsilon_2\int_0^T u_{\epsilon_1}^2(t)\,\mathrm{d}t.$$
$$(7)$$

This implies that

$$J_{\epsilon_1}(u_{\epsilon_1}) \leq J_{\epsilon_2}(u_{\epsilon_2}). \tag{8}$$

From (7), we obtain

$$J_{\epsilon_2}(u_{\epsilon_2}) - J_{\epsilon_1}(u_{\epsilon_2}) \leq J_{\epsilon_2}(u_{\epsilon_1}) - J_{\epsilon_1}(u_{\epsilon_1}),$$

and then

$$\int_0^T u_{\epsilon_2}^2(t)\,\mathrm{d}t \le \int_0^T u_{\epsilon_1}^2(t)\,\mathrm{d}t.$$

Thus $\|\chi_\omega z_{\epsilon_1}(T) - z_d\|_{L^2(\omega)}^2 \le \|\chi_\omega z_{\epsilon_2}(T) - z_d\|_{L^2(\omega)}^2$, which shows Statements 1 and 2 and the first part of Statement 3.

For $u = 0$, we have $z_u(T) = S(T)z_0$ and $\forall \epsilon > 0$,

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u_\epsilon(t)^2\,\mathrm{d}t$$
$$\le \|\chi_\omega S(T)z_0 - z_d\|_{L^2(\omega)}^2.$$

Then

$$0 \le \|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2$$
$$\le \|\chi_\omega S(t)z_0 - z_d\|_{L^2(\omega)}^2, \quad \forall \epsilon > 0.$$

Finally, $(\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)})_{\epsilon > 0}$ is bounded. Then we can extract a subsequence of $(\chi_\omega z_\epsilon(T) - z_d)_{\epsilon > 0}$ which converges weakly in $L^2(\omega)$. ∎

## 3. Regional minimum energy control problem

Here let us go back to the problem (2), and consider the set

$$R(T) = \bigcup_{u \in L^2[0,T]} \{z_u(T)\}$$

of the states reachable at time $T$ from $z_0$.

We have the main result.

**Theorem 2.** *Let $u_\epsilon$ be a solution of (3) and assume that $U_{ad}(\omega)$ is nonempty. Then we have*

$$u_\epsilon \to u^\star \quad \text{as } \epsilon \to 0 \text{ in } L^2[0,T]$$

*and*

$$\chi_\omega z_\epsilon \to \chi_\omega z_{u^\star} \quad \text{in } C([0,T];L^2(\omega)).$$

*Moreover, $u^\star$ is a solution to the problem (2).*

*Proof.* Using the optimality of $u_\epsilon$ for $J_\epsilon$, we have $\forall \epsilon > 0$, $\forall u \in L^2[0,T]$, $J_\epsilon(u_\epsilon) \le J_\epsilon(u)$, i.e.,

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u_\epsilon^2(t)\,\mathrm{d}t$$
$$\le \|\chi_\omega z_u(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u^2(t)\,\mathrm{d}t.$$

$U_{ad}(\omega)$ is nonempty, which means that there exists $v \in L^2[0,T]$ such that

$$\|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)}^2 = \min_{z \in R(T)} \|\chi_\omega z - z_d\|_{L^2(\omega)}^2.$$

Thus, we have

$$\int_0^T u_\epsilon^2(t)\,\mathrm{d}t \le \int_0^T u^2(t)\,\mathrm{d}t, \quad \forall u \in U_{ad}(\omega), \forall \epsilon > 0.$$
$$(9)$$

Therefore, we can extract a subsequence, also denoted by $(u_\epsilon)_{\epsilon > 0}$, such that $u_\epsilon \to u^\star$ weakly in $L^2[0,T]$ and $z_\epsilon \to z_{u^\star}$ strongly in $C([0,T];Z)$ as $\epsilon \to 0$ (see Ball *et al.*, 1982), and this implies that $\chi_\omega z_\epsilon \to \chi_\omega z_{u^\star}$ strongly in $C([0,T];L^2(\omega))$. Since $u_\epsilon \to u^\star$ weakly in $L^2[0,T]$, by the lower semi-continuity of the norm we have

$$\liminf_{\epsilon \to 0} \int_0^T u_\epsilon^2(t)\,\mathrm{d}t \ge \int_0^T u^{\star 2}(t)\,\mathrm{d}t \qquad (10)$$

and

$$\liminf_{\epsilon \to 0} J_\epsilon(u_\epsilon) \ge \|\chi_\omega z_{u^\star}(T) - z_d\|_{L^2(\omega)}^2.$$

Moreover, $J_\epsilon(u_\epsilon) \le J_\epsilon(u)$ $\forall u \in L^2[0,T]$, so

$$\limsup_{\epsilon \to 0} J_\epsilon(u_\epsilon)$$
$$\le \|\chi_\omega z_u(T) - z_d\|_{L^2(\omega)}^2 \quad \forall u \in L^2[0,T], \quad (11)$$

and, in particular,

$$\limsup_{\epsilon \to 0} J_\epsilon(u_\epsilon) \le \|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)}^2$$
$$\le \|\chi_\omega z_{u^\star}(T) - z_d\|_{L^2(\omega)}^2$$
$$\le \liminf_{\epsilon \to 0} J_\epsilon(u_\epsilon).$$

Hence

$$\lim_{\epsilon \to 0} J_\epsilon(u_\epsilon) = \lim_{\epsilon \to 0} \|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2$$
$$= \|\chi_\omega z_{u^\star}(T) - z_d\|_{L^2(\omega)}^2 \qquad (12)$$
$$= \|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)}^2.$$

Thus

$$\lim_{\epsilon \to 0} \|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 = \min_{z \in R(T)} \|\chi_\omega z - z_d\|_{L^2(\omega)}^2,$$

and $u^\star \in U_{ad}(\omega)$.

Furthermore,

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u_\epsilon^2(t)\,\mathrm{d}t$$
$$\le \|\chi_\omega z_{u^\star}(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u^{\star 2}(t)\,\mathrm{d}t.$$

From (12) it follows that

$$\int_0^T u_\epsilon^2(t)\,\mathrm{d}t \le \int_0^T u^{\star 2}(t)\,\mathrm{d}t, \quad \forall \epsilon > 0. \qquad (13)$$

Equations (10) and (13) show that

$$\int_0^T u_\epsilon^2(t)\,\mathrm{d}t \to \int_0^T u^{\star 2}(t)\,\mathrm{d}t \quad \text{as } \epsilon \to 0.$$

This result, together with the weak convergence of $(u_\epsilon)_{\epsilon>0}$ towards $u^\star$ in $L^2[0,T]$, implies that

$$\lim_{\epsilon \to 0} \int_0^T (u_\epsilon(t) - u^\star(t))^2 \,\mathrm{d}t = 0.$$

Using (9), we obtain

$$\int_0^T u^{\star 2}(t)\,\mathrm{d}t \le \int_0^T u^2(t)\,\mathrm{d}t, \quad \forall u \in U_{\mathrm{ad}}(\omega),$$

and hence $u^\star$ is a solution to the problem (2). ∎

**Remark 2.**

1. From the proof of Theorem 2, it follows that, if the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $L^2[0,T]$, then $U_{\mathrm{ad}}(\omega) \ne \emptyset$.

2. We do not give any result for the uniqueness except for the global case ($\omega = \Omega$). We have the following result.

**Proposition 3.** *Suppose that $U_{\mathrm{ad}}(\Omega)$ is nonempty and $L^2(\Omega)$ has an orthonormal basis $(\phi_n)_n$ of eigenfunctions of A. In addition, if A commutes with B, then the problem (2) has only one solution.*

*Proof.* First, the existence of a solution is ensured by Theorem 2. With no loss of generality, we may suppose that the eigenvalues of $A$ are simple. Now, $A$ and $B$ commute, so the mild solution of (1) can be written as

$$z_u(t) = S(t) \exp\Big(B \int_0^t u(s)\,\mathrm{d}s\Big) z_0,$$

where $\big(\exp\big(B \int_s^t u(r)\,\mathrm{d}r\big)\big)_{t\ge s}$ is the evolution operator generated by $uB$. For $z_0 \in L^2(\Omega)$, we have

$$z_u(t) = \sum_{n=1}^{+\infty} \exp \lambda_n t \langle \exp\big(B \int_0^t u(s)\,\mathrm{d}s\big) z_0, \phi_n \rangle \phi_n.$$

Then

$$z_u(T) - z_d$$
$$= \sum_{n=1}^{+\infty} \langle \exp(\lambda_n T) \exp\big(B \int_0^T u(s)\,\mathrm{d}s\big) z_0 - z_d, \phi_n \rangle \phi_n,$$

and

$$\| z_u(T) - z_d \|^2$$
$$= \sum_{n=1}^{+\infty} \langle \exp(\lambda_n T) \exp\big(B \int_0^T u(s)\,\mathrm{d}s\big) z_0 - z_d, \phi_n \rangle^2. \tag{14}$$

If $u$ and $v$ are two distinct solutions to the problem (2), then (14) implies

$$\int_0^T u(s)\,\mathrm{d}s = \int_0^T v(s)\,\mathrm{d}s.$$

The control $w = (u+v)/2$ lies in $U_{\mathrm{ad}}(\Omega)$, i.e.,

$$z_w(T) = S(T) \exp\Big(B \int_0^T \frac{1}{2}[u(s) + v(s)]\,\mathrm{d}s\Big) z_0$$
$$= z_u(T),$$

and

$$\|w\|_{L^2[0,T]}^2 = \frac{1}{4}\|u+v\|_{L^2[0,T]}^2$$
$$< \frac{1}{2}[\|u\|_{L^2[0,T]}^2 + \|v\|_{L^2[0,T]}^2]$$
$$= \|u\|_{L^2[0,T]}^2.$$

This contradiction implies that the minimum energy control is unique. ∎

**Remark 3.**

1. The above results remain true in the case of multi-controls, i.e., when the system is described by

$$\dot{z}(t) = Az(t) + \sum_{i=1}^p u_i(t) B_i z(t),$$

where $\forall i,\ 1 \le i \le p,\ u_i \in L^2[0,T]$, and $B_i$ is a bounded linear operator on $Z$.

2. In the same way we can solve the following general problem:

$$\begin{cases} \min \|u\|_{L^2[0,T]}^2 \\ \quad \text{with} \\ \langle \chi_\omega z_u(T) - z_d(T), G(\chi_\omega z_u(T) - z_d(T)) \rangle_{L^2(\omega)} \\ \\ + \int_0^T \langle \chi_\omega z_u(t) - z_d(t), Q(\chi_\omega z_u(t) - z_d(t)) \rangle\,\mathrm{d}t \\ \quad \text{minimum,} \end{cases} \tag{15}$$

where $z_d$ is a desired regular function.

The problem associated with (15) is

$$\begin{cases} \min \Phi_\epsilon(u), \\ u \in L^2[0,T] \end{cases} \tag{16}$$

with

$$\Phi_\epsilon(u)$$
$$= \langle \chi_\omega z_u(T) - z_d(T), G(\chi_\omega z_u(T) - z_d(T)) \rangle_{L^2(\omega)}$$
$$+ \int_0^T [\langle (\chi_\omega z_u(t) - z_d(t), Q(\chi_\omega z_u(t) - z_d(t)) \rangle_{L^2(\omega)}$$
$$+ \epsilon u^2(t)]\,\mathrm{d}t, \quad \epsilon > 0,$$

whose solution is given by

$$u(t) = -\frac{1}{\epsilon}\langle Bz(t), P(t)z(t) - U^*(T,t)\chi_\omega^* Gz_d(T)$$
$$- \int_t^T U^*(s,t)\chi_\omega^* Qz_d(s)\,\mathrm{d}s\rangle.$$

where $P$ is the self-adjoint and nonnegative operator solution of the equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle P(t)y, z\rangle + \langle P(t)y, (A+u(t)B)z\rangle \\ + \langle(A+u(t)B)y, P(t)z\rangle + \langle\chi_\omega^* Q\chi_\omega y, z\rangle = 0, \\ P(T) = \chi_\omega^* G\chi_\omega, \text{ where } y, z \in D(A). \end{cases}$$

3. If $z_d(\cdot)$ is exactly reachable with the control $v$, then

$$u_\epsilon \to v \qquad \text{in } L^2[0,T] \text{ strongly},$$

$$\chi_\omega z_\epsilon \to z_d \qquad \text{in } C([0,T]; L^2(\omega)) \text{ strongly},$$

as $\epsilon \to 0$, where $u_\epsilon$ is a control which minimizes in $L^2[0,T]$ the quadratic cost

$$J_\epsilon(u)$$
$$= \|\chi_\omega z_u(T) - z_d(T)\|_{L^2(\omega)}^2$$
$$+ \int_0^T [\langle\chi_\omega z_u(t) - z_d(t), \chi_\omega z_u(t) - z_d(t)\rangle_{L^2(\omega)}$$
$$+ \epsilon u^2(t)]\,\mathrm{d}t, \qquad \epsilon > 0.$$

We now deal with the case where $U_{\mathrm{ad}}(\omega)$ is an empty set.

**Theorem 3.** *Suppose that $U_{\mathrm{ad}}(\omega)$ is empty. Then*

$$\lim_{\epsilon \to 0} \|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 = \inf_{z \in R(T)} \|\chi_\omega z - z_d\|_{L^2(\omega)}^2.$$

*Proof.* Let

$$F = \{\|\chi_\omega z - z_d\|_{L^2(\omega)} \mid z \in R(T)\}.$$

Then, $F$ is a nonempty subset of $\mathbb{R}^+$. Therefore, $F$ has a lower bound denoted by $a$. According to Proposition 1, $(J_\epsilon(u_\epsilon))_{\epsilon > 0}$ is a decreasing sequence as $\epsilon \to 0$, and $J_\epsilon(u_\epsilon) \geq 0, \forall \epsilon > 0$.

Hence, it converges in $\mathbb{R}$ towards a limit denoted by $J$. Similarly, $(\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)})_{\epsilon > 0}$ is a nonnegative and decreasing sequence. Thus, as $\epsilon \to 0$ it converges in $\mathbb{R}$ towards a limit denoted by $b$. Let us show that $b = a$. Suppose that $b > a$. Then there exists $v \in L^2[0,\ T]$ such that

$$a < \|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)} < b. \tag{17}$$

Now,

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T u_\epsilon^2(t)\,\mathrm{d}t$$
$$\leq \|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)}^2 + \epsilon \int_0^T v^2(t)\,\mathrm{d}t. \tag{18}$$

Equations (17) and (18) imply that

$$\int_0^T u_\epsilon^2(t)\,\mathrm{d}t \leq \int_0^T v^2(t)\,\mathrm{d}t.$$

Thus, according to Remark 2, $U_{\mathrm{ad}}^T(\omega)$ is nonempty, which is a contradiction. ∎

**Remark 4.**

1. The family of controls $(u_\epsilon)_{\epsilon > 0}$ is not bounded in $L^2[0,T]$ (Remark 2) and for a fixed $\epsilon$ and for all $\chi_\omega z_v(T)$ such that

$$\|\chi_\omega z_\epsilon(T) - z_d\|_{L^2(\omega)} = \|\chi_\omega z_v(T) - z_d\|_{L^2(\omega)}$$

according to (18) we have

$$\int_0^T u_\epsilon^2(t)\,\mathrm{d}t \leq \int_0^T v^2(t)\,\mathrm{d}t.$$

2. The approach used to solve the optimal control problem assumes a bounded control operator. However, the unbounded case may be carried out in a similar manner taking more regular controls which allow regular system states. This means that the control is selected such that the state $z$ is in $Z = L^2(\Omega)$.

## 4. Numerical approach and simulations

We have seen that, if an optimal control solution to the problem (2) exists, such a control may be approximated by the solution $u_\epsilon$ to the problem (3), which in turn may be implemented by the following formula:

$$\begin{cases} u_{n+1}(t) = -n\langle Bz_n(t), P_n(t)z_n(t) - U_n^*(T,t)\chi_\omega^* z_d\rangle, \\ u_0 = 0, \end{cases} \tag{19}$$

where $P_n$ is the selfadjoint and nonnegative operator solution of the Riccati equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\langle P_n(t)y, z\rangle + \langle P_n(t)y, (A+u_n(t)B)z\rangle \\ + \langle(A+u_n(t)B)y, P_n(t)z\rangle = 0, \\ P_n(T) = \chi_\omega^* \chi_\omega \text{ with } y, z \in D(A), \end{cases} \tag{20}$$

whose solution can be achieved by the algorithm given by El Alami (1988).

This allows us to consider the following algorithm:

*Step 1:* Initialize system data: $z_0$, $u_0 = 0$, desired state $z_d$, threshold accuracy $\varepsilon$, subregion $\omega$ and sensor location $b$.

*Step 2:* Until $\| u_{n+1} - u_n \| \leq \varepsilon$ repeat

$\ominus$ Solve Eqn. (20) which gives $P_n$.
$\ominus$ Solve Eqn. (1) which gives $z_n(t)$.
$\ominus$ Compute $u_{n+1}$ by the formula (19).

The control $u_n$ steers the system to the desired state $z_d$ at time $T$.

To illustrate the above algorithm, consider the following examples.

**Example 1.** Let $\Omega = ]0, 1[$ and consider the bilinear system described by the following evolution equation:

$$
\begin{cases}
\dfrac{\partial z}{\partial t}(x,t) = \alpha \dfrac{\partial^2 z(x,t)}{\partial x^2} + \beta z(x,t) \\
\qquad\qquad + \gamma u(t) z(x,t) & \text{in } \Omega \times ]0, T[, \\
z(x,0) = z_0(x) & \text{in } \Omega, \\
z(0,t) = z(1,t) = 0 & \text{on } ]0, T[,
\end{cases}
\tag{21}
$$

where $\alpha$, $\beta$ and $\gamma$ are positive real numbers. This equation may represent a simplified model of the temperature distribution in a furnace.

The system (21) looks like (1) with

$$
\tilde{A} = \alpha \frac{\partial^2}{\partial x^2} + \beta
$$

with the domain

$$
D(\tilde{A}) = \left\{ z \in H^2(0,1) \mid z(0) = z(1) = 0 \right\}.
$$

The operator $\tilde{A}$ admits a set of eigenfunctions $\phi_i(\cdot)$ associated with the eigenvalues $\lambda_i$ given by

$$
\phi_i(x) = \sqrt{2}\sin(i\pi x), \quad \lambda_i = \beta - \alpha i^2 \pi^2, \quad i \geq 1.
$$

The solution (21) is approximated by

$$
z(x,t) \simeq \sum_{i=1}^{M} a_i(t)\phi_i(x).
$$

Let $z_0(x) = \sin(\pi x), z_d(x) = 8x(1-x), \alpha = 0.01, \beta = 0.01, \gamma = 0.02, \varepsilon = 0.0001$ and $T = 1$. Augmenting the truncation order $M$ beyond 5 does not improve the simulation results.

Using the above algorithm for different regions of $\omega$ and after the 7-th iteration, we have

(i) Case of $\omega = ]0.4, 0.6[$: see Figs. 1 and 2,
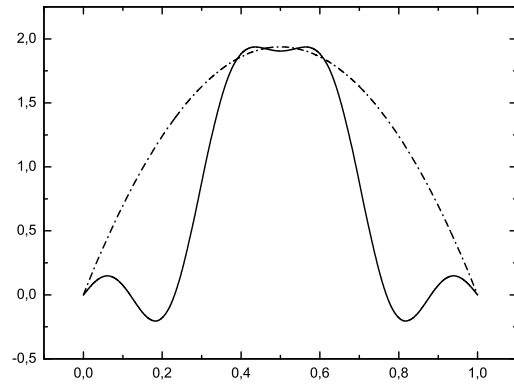
(ii) Case of $\omega = ]0.6, 1[$: see Figs. 3 and 4.

♦



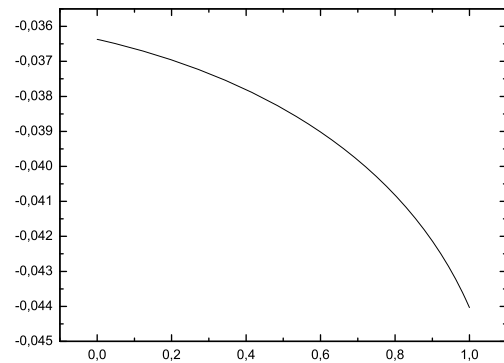Fig. 1. Desired state $z_d$ (dashed line) and the reached state $z_{u_7}(T)$ (continuous line) in $\omega$.



Fig. 2. Optimal control function $u^\star(\cdot) \simeq u_7(\cdot)$.

(iii) Case of $\omega = ]0.8, 1[$: see Figs. 5 and 6.

**Example 2.** Let us consider the bilinear system with the domain $\Omega = ]0, 1[$ described by the following equation:

$$
\begin{cases}
\dfrac{\partial z}{\partial t}(x,t) = \alpha \dfrac{\partial^2 z(x,t)}{\partial x^2} + \beta z(x,t) + \tilde{\gamma} u(t) z(x,t) \\
\qquad\qquad + \delta(x-b)u(t) & \text{in } \Omega \times ]0, T[, \\
\\
z(x,0) = z_0(x) & \text{in } \Omega, \\
\\
z(0,t) = z(1,t) = 0 & \text{on } ]0, T[.
\end{cases}
\tag{22}
$$

The operator

$$
\tilde{A} = \alpha \frac{\partial^2}{\partial x^2} + \beta
$$

has the domain $D(\tilde{A}) = \{ z \in H^2(0,1) \mid z(0) = z(1) = 0 \}$ and $\delta$ is the Dirac delta. Let $z_0(x) = 6.4x(1-x), \alpha = 0.01, \beta = 0.01, \tilde{\gamma} = 0.02, \varepsilon = 10^{-4}, T = 1$ and $b = 0.1$.
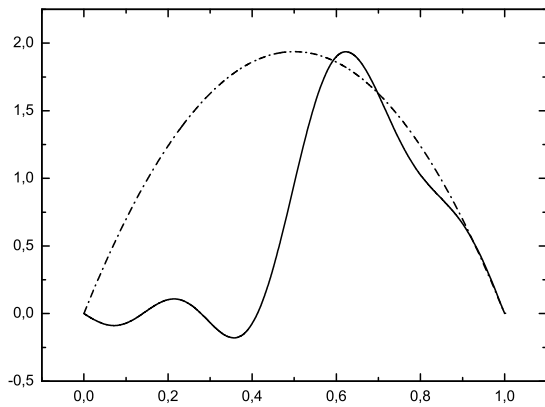
For

$$
\omega = [0.35, 0.65]
$$

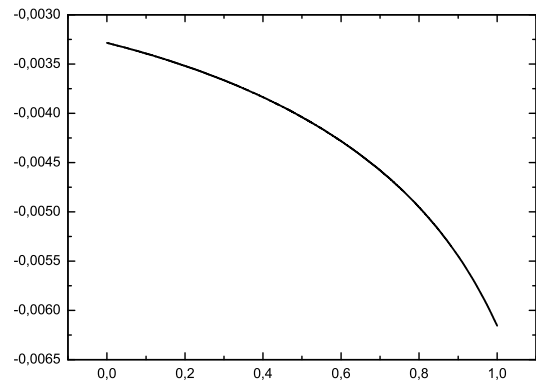Fig. 3. Desired state $z_d$ (dashed line) and the reached state $z_{u_7}(T)$ (continuous line) in $\omega$.
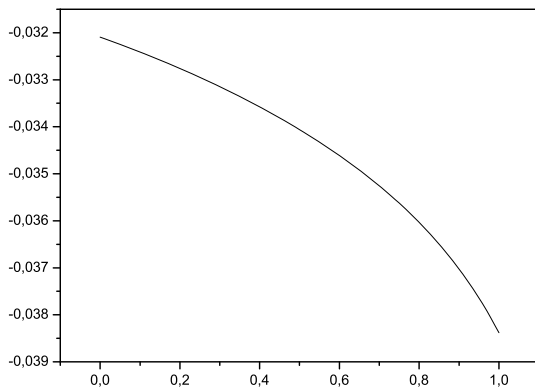


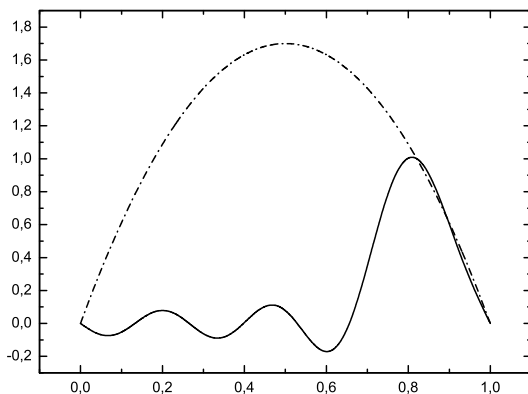Fig. 4. Optimal control function $u^\star(\cdot) \simeq u_7(\cdot)$.



Fig. 5. Desired state $z_d$ (dashed line) and the reached state $z_{u_7}(T)$ (continuous line) in $\omega$.

and

$$z_d = \begin{cases} 1.5 + 300(x - 0.3)^2 \\ \quad \times (x - 0.7)^2 & \text{if } x \in [0.35, 0.65], \\ 0 & \text{otherwise,} \end{cases}$$



Fig. 6. Optimal control function $u^\star(.) \simeq u_7(.)$.

application of the above algorithm gives the results presented in Figs. 7 and 8.

## 5. Conclusion

A regional controllability problem for bilinear systems was considered and an optimal control was characterized. Under adding conditions, the uniqueness of such a control was proved. Moreover, a numerical approach was developed based on a quadratic control problem. The obtained results were successfully tested through numerical examples and simulations. Many questions remain still open, e.g., the extension of the present results to a boundary subregion. The case of systems with time delays would also be very interesting.

## Acknowledgment

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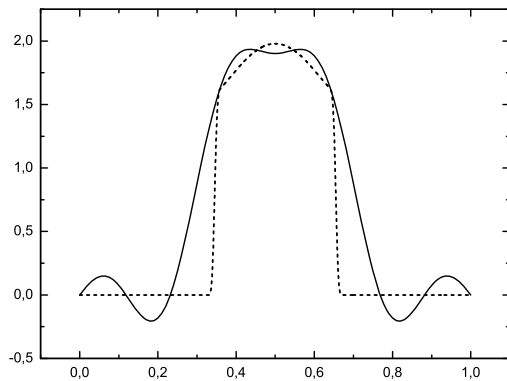
Fig. 7. Desired state $z_d$ (dashed line) and the reached state $z_{u_6}(T)$ (continuous line) in $\omega$.
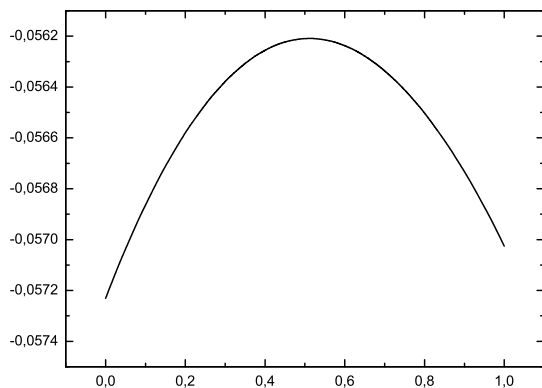


Fig. 8. Optimal control function $u^\star(\cdot) \simeq u_6(\cdot)$.

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