Int. J. Appl. Math. Comput. Sci., *2012, Vol. 22, No. 2, 269–280*
DOI: 10.2478/v10006-012-0020-y

TOPOLOGY OPTIMIZATION OF QUASISTATIC CONTACT PROBLEMS

ANDRZEJ MYŚLIŃSKI

Systems Research Institute
Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland
e-mail: myslinsk@ibspan.waw.pl

This paper deals with the formulation of a necessary optimality condition for a topology optimization problem for an elastic contact problem with Tresca friction. In the paper a quasistatic contact model is considered, rather than a stationary one used in the literature. The functional approximating the normal contact stress is chosen as the shape functional. The aim of the topology optimization problem considered is to find the optimal material distribution inside a design domain occupied by the body in unilateral contact with the rigid foundation to obtain the optimally shaped domain for which the normal contact stress along the contact boundary is minimized. The volume of the body is assumed to be bounded. Using the material derivative and asymptotic expansion methods as well as the results concerning the differentiability of solutions to quasistatic variational inequalities, the topological derivative of the shape functional is calculated and a necessary optimality condition is formulated.

Keywords: quasistatic contact problem, elasticity, Tresca friction, topology optimization.

1. Introduction

Consider a domain $\Omega \subset \mathbb{R}^2$ occupied by a body or a structure and the solution $u = u(\Omega)$ of a system of partial differential equations defined in Ω and describing the state of the body. The aim of topology optimization is to find an optimal distribution of the body material within the geometrical domain Ω resulting in its optimal shape in the sense of some shape functional (Allaire *et al.*, 2004; Amstuz *et al.*, 2008; Garreau *et al.*, 2001; Sokołowski *et al.*, 1999; 2004). Unlike in the case of classical shape optimization (Haslinger and Mäkinen, 2003; Jarušek *et al.*, 2003; Myśliński, 2006; Sokołowski and Zolesio, 1992) based on domain boundary perturbations only, the topology of the domain Ω occupied by the body may change through the nucleation of small holes or the inclusion of weak material.

A classical approach to topology optimization problems is based on relaxed formulations and the homogenization method (Allaire, 2002). The obtained optimal solution is a quasi-uniform distribution of composite materials, rather than a classical design (Garreau *et al.*, 2001). The density approach, also called the SIMP (Solid Isotropic Material with Penalization) method (Bendsoe *et al.*, 2003), is another currently used topology optimization method. It consists in using a fictitious isotropic material whose elasticity tensor is assumed to be a function of penalized material density, represented by an exponent parameter. The SIMP method has been used by Strömberg and Klabring (2010) to solve numerically a topology optimization problem for an elastic structure with unilateral boundary conditions.

In recent years the topological derivative method (Garreau *et al.*, 2001; Kowalewski *et al.*, 2010; Nazarov and Sokołowski, 2003; Novotny *et al.*, 2005; Sokołowski and Żochowski, 1999; 2004) has emerged as an attractive alternative to analyze and solve numerically topology optimization problems, especially of elastic structures, without employing the homogenization approach. The topological derivative gives an indication on the sensitivity of the shape functional with respect to the nucleation of a small hole or a cavity, or more generally a small defect in a geometrical domain Ω around a given point. This concept of topological sensitivity analysis was introduced in the field of shape optimization by Eschenauer *et al.* (1994) and was first mathematically justified by Garreau *et al.* (2001) as well as Sokołowski and Żochowski (1999). The modern mathematical background for evaluation of topological derivatives by asymptotic analysis techniques of boundary value problems is established by Nazarov and Sokołowski (2003).

In the literature, most papers (see Chambolle, 2003;

Denkowski and Migórski, 1998; Eschenauer *et al.*, 1994; Fulmański *et al.*, 2007; Garreau *et al.*, 2001; Myśliński, 2008; 2010; Nazarov and Sokołowski, 2003; Novotny *et al.*, 2005; Sokołowski and Zolesio, 1999; Sokołowski and Żochowski 2004; 2005; 2008; Strömberg and Klabring, 2010) are devoted to asymptotic and topology sensitivity analysis for elliptic boundary value problems. A few papers only (e.g., Amstuz *et al.*, 2008; Kowalewski *et al.*, 2010) address this issue for the shape functionals depending on a solution to time-dependent boundary value problems. One of the reasons is that the approaches useful for stationary boundary value problems fail for evolution problems (Kowalewski *et al.*, 2010).

The approach of Amstuz *et al.* (2008) extends the ideas of Garreau *et al.* (2001). Material occupying the integration domain is assumed to consist of the strong material and the weak material occupying the holes. A polarization matrix is used to calculate the topological derivatives of the different shape functionals depending on solutions to heat or wave equations. The paper by Kowalewski *et al.* (2010) deals with the sensitivity analysis of the optimal control problem for the wave equation with respect to the small hole or cavity in the geometrical domain using the "hidden regularity" argument for boundary traces as well as the expansion of the elliptic Steklov–Poincaré operator. Evolution boundary value problems may be also considered as the eigenvalue problem. Using the single layer potential technique and the polarization matrix approach, in the monograph by Ammari *et al.* (2009) asymptotic expansions to solutions of eigenvalue problems have been provided.

The frictional contact phenomenon between deformable bodies occurs frequently in industry or everyday life. It happens, among others, between the surfaces of braking pads and wheels, the tire and the road, the piston and the shirt or a shoe and the floor. Since the frictional contact leads to softening and possible damage of the contacting surfaces, the prediction and control of the evolution of frictional contact processes is the subject of intensive research. The mathematical or engineering literature concerning this topic is rather extensive (see the references in the monographs of Eck *et al.* (2005) as well as Han and Sofonea (2002)). Asymptotic and topology sensitivity analysis of solutions to unilateral stationary boundary value problems in elasticity is performed by Fulmański *et al.* (2007), Myśliński (2010) and Sokołowski *et al.* (2005; 2008). Simultaneous shape and topology optimization of elastic structures where both the boundary perturbation and the nucleation of holes inside the domain occur is considered, among others, by Myśliński (2008) as well as Sokołowski and Żochowski (2004). The main difficulty in topology sensitivity analysis of contact problems is associated with the nonlinearity of the non-penetration condition over the contact zone, which makes this boundary value problem non-smooth.

This paper is concerned with the application of a topological derivative approach to formulate a necessary optimality condition for a structural optimization problem for elliptic contact problems with Tresca friction. Unlike in the previous works of Myśliński (2008; 2010), where the stationary contact model is used, here the quasistatic contact problem is considered. Quasistatic processes arise when the external forces applied to a system vary slowly in time. This means that the system is observed over a long-time scale and the acceleration is negligible. The key difference between static and quasistatic contact problems is the dependence of the friction on the sliding velocity, rather than on the displacement. The quasistatic contact problem in elasticity is governed by the elliptic variational inequality where displacement and stress fields are time-dependent. The existence of solutions to different quasistatic or dynamic hemivariational inequalities is shown by Ayyad *et al.* (2007), Duvaut and Lions (1972), Denkowski and Migórski (1998), Eck *et al.* (2005), Han and Sofonea (2002) as well as Rocca and Cocu (2001). Numerical methods for solving contact problems are discussed by Haslinger and Mäkinen (2003) as well as Hüber *et al.* (2008).

The paper is organized as follows. Section 2 deals with the formulation of the quasistatic contact problem as well as the structural optimization problem. The goal of this optimization problem is to find a distribution of the body material within the geometrical domain occupied by the body in unilateral contact with the rigid foundation which would ensure the minimum value of the shape functional describing the normal contact stress. The volume of the body is bounded. The paper is confined to topology optimization only, i.e., the domain occupied by the body is subject to topology variation only while the external boundary of this domain is assumed not to be perturbed. A topological derivative formula of the domain functional is calculated using the shape derivative (Sokołowski and Zolesio, 1992; Sokołowski and Żochowski, 2004) and asymptotic expansion (Sokołowski and Żochowski, 1999) methods. The calculated topological derivative is employed to formulate in Section 3 the necessary optimality condition.

We shall use the following notation: $\Omega \subset \mathbb{R}^2$ will denote a bounded domain with a Lipschitz continuous boundary Γ. The time variable will be denoted by t and the time interval by $I = (0, \mathcal{T}), 0 < \mathcal{T} < \infty$. By $H^k(\Omega)$, $k \in (0, \infty)$, we will denote the Sobolev space of functions defined on Ω and having derivatives in all directions of the order k and belonging to $L^2(\Omega)$ (Han and Sofonea, 2002). For an interval I and a Banach space B^{sp}, $L^p(I; B^{sp})$, $p \in (1, \infty)$ denotes the usual Bochner space (Eck *et al.*, 2005). $\dot{u} = \mathrm{d}u/\mathrm{d}t$ will denote the first order derivative of the function u with respect to t. $\dot{u}_N$ and $\dot{u}_T$ will denote normal and tangential components of the function $\dot{u}$, respectively. $\mathcal{S}_2$ denotes the space of the

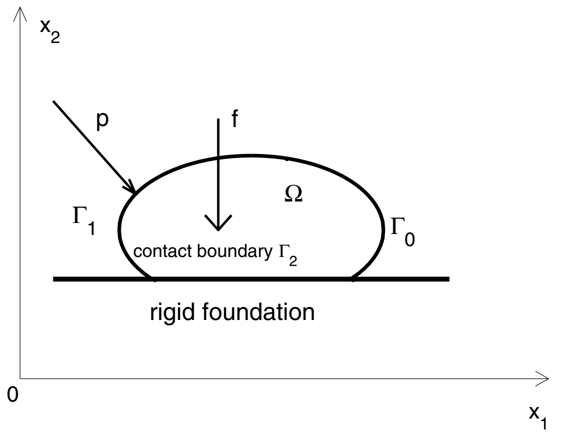

Fig. 1. Initial domain Ω.

second order symmetric tensors (Han and Sofonea, 2002). The dot product of two vectors $w, z \in \mathbb{R}^d$ is defined as

$$w \cdot z = \sum_{i=1}^{d} w_i z_i.$$

2. Problem formulation

Consider deformations of an elastic body occupying a domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz continuous boundary Γ (see Fig. 1). Let $S \subset \mathbb{R}^2$ and $D \subset \mathbb{R}^2$ denote given bounded domains. The so-called *hold-all domain D* is assumed to possess a piecewise smooth boundary. Domain Ω is assumed to belong to the set O_l defined as follows:

$$O_l = \{\Omega \subset \mathbb{R}^2 : \Omega \text{ is open}, \quad S \subset \Omega \subset D, \ {}^{\#}\Omega^c \le l\}, \tag{1}$$

where ${}^{\#}\Omega^c$ denotes the number of connected components of the complement Ω^c of Ω with respect to D and $l \ge 1$ is a given integer. Moreover, all perturbations $\delta\Omega$ of Ω are assumed to satisfy $\delta\Omega \in O_l$.

The boundary Γ is partitioned into three open measurable disjoint parts $\Gamma_0, \Gamma_1, \Gamma_2$ such that $\mathrm{meas}(\Gamma_0) > 0$ and $\Gamma_i \cap \Gamma_j = \emptyset$, $i \ne j$, $i, j = 0, 1, 2$, as well as $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. Let $0 < \mathcal{T} < \infty$ and let $I = (0, \mathcal{T})$ denote the time interval of interest. Assume the body is loaded in $\Omega \times I$ by the body force $f(x,t) = (f_1(x,t), f_2(x,t))$, $(x,t) \in \Omega \times I$. Moreover, on $\Gamma_1 \times I$ acts a surface traction $p(x,t) = (p_1(x,t), p_2(x,t))$, $(x,t) \in \Gamma_1 \times I$. The body is clamped on $\Gamma_0 \times I$. The contact between the bodies may occur on $\Gamma_2 \times I$. Denote by $u(x,t) = (u_1(x,t), u_2(x,t))$, $(x,t) \in \Omega \times I$ a displacement of the body and by $\sigma(x,t) = \{\sigma_{ij}(u(x,t))\}$, $i, j = 1, 2$, the stress field in the body. Moreover, the notation $u(t) = u(x,t)$ will be used to emphasize the dependence of u on t.

We shall consider elastic bodies obeying Hooke's law (Ayyad *et al.*, 2007; Eck *et al.*, 2005; Han and Sofonea, 2002):

$$\sigma_{ij}(u(x,t)) = c_{ijkl}(x) e_{kl}(u(x,t)), \quad i, j, k, l = 1, 2, \quad (x,t) \in \Omega \times I. \tag{2}$$

It is assumed that components $c_{ijkl}(x)$, $i, j, k, l = 1, 2$, of the elasticity tensor satisfy (Duvaut and Lions, 1972; Eck *et al.*, 2005; Han and Sofonea, 2002) usual symmetry, boundedness and ellipticity conditions, i.e.,

$$c_{ijkl}(x) \in L^{\infty}(\Omega), \quad c_{ijkl} = c_{jikl} = c_{klij}, \tag{3}$$

$$\exists \alpha_1 > 0, \ \alpha_0 > 0 :$$

$$\alpha_0 t_{ij} t_{ij} \le c_{ijkl}(x) t_{ij} t_{kl} \le \alpha_1 t_{ij} t_{kl}, \tag{4}$$

for almost all $x \in \Omega$, for all symmetric 2×2 matrices $t_{ij} \in \mathcal{S}_2$, $i, j = 1, 2$, with constants $0 < \alpha_0 \le \alpha_1$. We use here and throughout the paper the summation convention over repeated indices (Duvaut and Lions, 1972; Eck *et al.*, 2005; Han and Sofonea, 2002; Haslinger and Mäkinen, 2003).

We shall also assume that in the neighbourhood $\mathcal{N}$ of the boundary Γ_2 the components $c_{ijkl}(x)$, $i, j, k, l = 1, 2$, are more regular, i.e.,

$$c_{ijkl}(x) \in C^{0,\beta}(\mathcal{N}), \quad 0 < \beta < \frac{1}{2}. \tag{5}$$

The strain $e_{kl}(u(x,t))$, $k, l = 1, 2$, is defined by

$$e_{kl}(u(x,t)) = \frac{1}{2}(u_{k,l}(x,t) + u_{l,k}(x,t)),$$

$$u_{k,l}(x,t) = \frac{\partial u_k(x,t)}{\partial x_l}. \tag{6}$$

Consider a quasistatic evolution contact process. Under the previous assumptions the formulation of the contact problem is the following: Find a displacement $u : \Omega \times \bar{I} \to \mathbb{R}^2$ and a stress field $\sigma : \Omega \times \bar{I} \to \mathcal{S}_2$ such that (Duvaut and Lions, 1972; Han and Sofonea, 2002)

$$-\sigma_{ij}(u)_{,j} = f_i(x,t) \quad \text{in} \quad \Omega \times I, \quad i, j = 1, 2, \tag{7}$$

where

$$\sigma_{ij}(u)_{,j} = \sigma_{ij}(u(x,t))_{,j} = \frac{\partial \sigma_{ij}(u(x,t))}{\partial x_j}, \quad i, j = 1, 2.$$

The following initial condition is imposed:

$$u_i(0,x) = u_{0i}, \quad i = 1, 2, \ \text{in} \ \Omega, \tag{8}$$

where u_{0i} are given functions. The boundary conditions have the form

$$u_i(x,t) = 0 \qquad \text{on } \Gamma_0 \times I, \qquad i = 1, 2, \tag{9}$$

$$\sigma_{ij}(u) n_j = p_i(x,t) \qquad \text{on } \Gamma_1 \times I, \quad i, j = 2, \tag{10}$$

$$u_N \le 0, \quad \sigma_N \le 0, \quad u_N\sigma_N = 0, \quad \text{on } \Gamma_2 \times I, \tag{11}$$

$$|\sigma_T| < \mathcal{F}|\sigma_N| \quad \Rightarrow \quad \dot{u}_T = 0 \ \text{ on } \Gamma_2 \times I, \tag{12}$$

$$|\sigma_T| = \mathcal{F}|\sigma_N| \quad \Rightarrow$$
$$\exists \lambda \ge 0 : \quad \dot{u}_T = -\lambda \sigma_T \quad \text{on } \Gamma_2 \times I, \tag{13}$$

where $n = (n_1, n_2)$ is the outward unit normal vector to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ on the boundary Γ, respectively. The tangential components of displacement u and stress σ on the boundary Γ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. The notation $|\cdot|$ signifies the absolute value when applied to a scalar and the Euclidean norm when applied to an element of $\mathbb{R}^d$, $d \ge 2$.

Recall from the work of Han and Sofonea (2002) that the non-penetration condition (11) means that if $u_N = 0$, the surface of the body is in contact with the surface of the rigid foundation which exerts a normal compression force $\sigma_N < 0$ on the body. If $u_N < 0$, there is no contact between the body and the foundation and consequently the foundation does not produce a reaction force towards the body, i.e., $\sigma_N = 0$. The conditions (12) and (13) describe the Coulomb friction law. At any moment the contact boundary is divided by these conditions into two zones: the stick zone and the slip zone. $\mathcal{F} \ge 0$ denotes a friction coefficient. Note that the contact problem (7)–(13) is time dependent due to the dependence of the body forces and the surface tractions on time as well as to the formulation of the Coulomb friction law (12)–(13) in terms of sliding velocity.

2.1. Variational formulation. Consider the contact problem (7)–(13) in the variational form. Let us introduce the space of virtual displacements,

$$F = \{z \in H^1(\Omega; \mathbb{R}^2) \ : \ z_i = 0 \ \text{ a.e. on } \Gamma_0, \ i = 1, 2\}, \tag{14}$$
$$W = H^1(I; F), \tag{15}$$

and the set of kinematically admissible displacements,

$$K = \{z \in F \ : \ z_N \le 0 \quad \text{a.e. on } \Gamma_2 \}, \tag{16}$$

as well as the bilinear form: $a(\cdot, \cdot) \ : \ F \times F \to \mathbb{R}$, given by

$$a(u, v) = \int_\Omega c_{ijkl} e_{ij}(u) e_{kl}(v) \, \mathrm{d}x. \tag{17}$$

The space $L^2(\Omega; \mathbb{R}^2)$ and the Sobolev spaces $H^1(\Omega; \mathbb{R}^2)$, $H^1(I; (L^2(\Omega; \mathbb{R}^2))$ as well as $L^2(\Gamma_1; \mathbb{R}^2)$, $H^{-1/2}(\Gamma_1; \mathbb{R}^2)$ are defined by Allaire (2002), Duvaut and Lions (1972) as well as Eck *et al.* (2005). The conditions (4) and (9) imply that the bilinear form (17) is continuous and coercive (Eck *et al.*, 2005), i.e.,

$$\exists \tilde{M} > 0 : |a(u, v)| \le \tilde{M} \parallel u \parallel_{H^1(\Omega;\mathbb{R}^2)} \parallel v \parallel_{H^1(\Omega;\mathbb{R}^2)}, \tag{18}$$
$$\forall (u, v) \in H^1(\Omega; \mathbb{R}^2) \times H^1(\Omega; \mathbb{R}^2),$$
$$\exists \tilde{m} > 0 : \quad a(v, v) \ge \tilde{m} \parallel v \parallel^2_{H^1(\Omega;\mathbb{R}^2)}, \quad \forall v \in F. \tag{19}$$

Let us assume that

$$f \in H^1(I; L^2(\Omega; \mathbb{R}^2)), \tag{20}$$
$$p \in H^1(I; L^2(\Gamma_1; \mathbb{R}^2)), \tag{21}$$
$$\mathcal{F} \ge 0, \quad \mathcal{F} \in C^1(\Gamma_2; \mathbb{R}), \tag{22}$$
$$\parallel \mathcal{F} \parallel_{L^\infty(\Gamma_2)} \le \sqrt{\frac{\alpha_0}{2\tilde{M}}}, \tag{23}$$
$$\parallel \mathcal{F} \parallel_{H^{1/2}(\Gamma_2)} \le \frac{\tilde{m}}{\tilde{M} \parallel \gamma \parallel_0 \parallel \tilde{\gamma} \parallel_1} \tag{24}$$

are given. $\parallel \gamma \parallel_0$ and $\parallel \tilde{\gamma} \parallel_1$ denote the norms (Eck *et al.*, 2005; Han and Sofonea, 2002; Rocca and Cocu, 2001) of the trace mapping $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ and the linear bounded extension mapping $\tilde{\gamma} : H^{1/2}(\Gamma) \to H^1(\Omega)$, respectively. Moreover, we assume, that the initial displacement $u(0) = u_0 = \{u_{0i}\}_{i=1}^2$ belongs to K and satisfies the compatibility condition

$$a(u_0, v - u_0) + \int_{\Gamma_2} \mathcal{F}|\sigma_N|(|v_T| - |u_{0T}|) \, \mathrm{d}s \ge \int_\Omega f(v - u_0) \mathrm{d}x + \int_{\Gamma_1} p(v - u_0) \, \mathrm{d}s, \quad \forall v \in K. \tag{25}$$

The problem (7)–(13) is equivalent to the following variational problem (Duvaut and Lions, 1972; Eck *et al.*, 2005): Find $u \in W$ such that $u(0) = u_0$ in Ω and $u(t) \in K$ for almost all $t \in I$ and satisfying the following system:

$$a(u, v - \dot{u}) + \int_{\Gamma_2} \mathcal{F}|\sigma_N|(|v_T| - |\dot{u}_T|) \, \mathrm{d}s \ge \int_\Omega f \cdot (v - \dot{u}) \, \mathrm{d}x + \int_{\Gamma_1} p \cdot (v - \dot{u}) \, \mathrm{d}s, \quad \forall v \in K. \tag{26}$$

The existence of solutions to the system (26) is shown by Rocca and Cocu (2001),

Theorem 1. *Assume the following:*

(i) the data are smooth enough, i.e., the conditions (3)–(5), (20), (22) and (25) are satisfied;

(ii) the boundary Γ_2 *is* $C^{1,\beta}$, $0 < \beta < 1/2$*;*

(iii) the friction coefficient $\mathcal{F}$ *is small enough, i.e., it satisfies (24).*

Then there exists at least one solution $u \in W$ to the problem (26).

Proof. It is based on the discretization of the variational inequality (26) with respect to time using an implicit scheme as well as the backward finite difference approximation of the time derivative $\dot{u}$. Moreover, the friction functional is regularized using a convex differentiable function. The existence of the solution to the discretized problem is shown introducing an auxiliary contact problem with a given friction as well as using Schauder's fixed point theorem. Taking the limit of the sequence of solutions to discretized contact problems as the friction regularization parameter as well as the time discretization parameter tend to zero, the existence of a solution to the original quasistatic problem (26) is shown. For details of the proof, see the work of Rocca and Cocu (2001). ■

Note that, from Theorem 1 as well as from (7), (20), (21), (25) we also deduce the existence of the stress field $\sigma(u(t))$ given by (2) and corresponding to the solution $u \in W$ of the system (26). This stress field $\sigma \in H^1(I;\mathcal{H})$, where $\mathcal{H} = \{\tau_{ij} \in L^2(\Omega;\mathbb{R}^4) : \tau_{ij} = \tau ji, \quad \tau_{ij,j} \in L^2(\Omega;\mathbb{R}^4), \; i,j = 1,2\}$.

In order to ensure the uniqueness of the solution to the variational inequality (26), we confine ourselves to considering the quasistatic contact problem with a prescribed friction (of the Tresca type), i.e.,

$$|\sigma_T| \leq g, \tag{27}$$

where $g \geq 0$, $g \in L^\infty(\Gamma_2)$, represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. From now on, without loss of generality, we assume $g = 1$. The conditions (12)–(13) are replaced by the following (Han and Sofonea, 2002; Haslinger and Mäkinen, 2003):

$$\dot{u}_T\sigma_T + |\dot{u}_T| = 0, \quad |\sigma_T| \leq 1 \quad \text{on } \Gamma_2 \times \mathrm{I}. \tag{28}$$

Introducing the sets

$$\Lambda = \{\lambda \in L^2(\Gamma_2;\mathbb{R}^2) \; : \; |\lambda| \leq 1 \quad \text{a.e. on } \Gamma_2 \}, \tag{29}$$

$$\tilde{\Lambda} = H^1(I;\Lambda), \tag{30}$$

and taking into account (28), the system (26) takes the following form: Find $(u,\lambda) \in W \times \tilde{\Lambda}$ such that $(u(t),\lambda(t)) \in K \times \Lambda$, $u(0)$ satisfies (25) with the Tresca friction law (27), rather than the Coulomb law (12)–(13), $\lambda(0) \in \Lambda$ and

$$a(u, v - \dot{u}) + \int_{\Gamma_2} \lambda \cdot (v_T - \dot{u}_T)\,\mathrm{d}x \geq \int_\Omega f \cdot (v - \dot{u})\,\mathrm{d}x + \int_{\Gamma_1} p \cdot (v - \dot{u})\,\mathrm{d}x, \quad \forall v \in K, \tag{31}$$

$$\int_{\Gamma_2} \lambda \cdot \dot{u}_T\,\mathrm{d}s \leq \int_{\Gamma_2} \zeta \cdot \dot{u}_T\,\mathrm{d}s \quad \forall \zeta \in \Lambda. \tag{32}$$

From Theorem 1 and the works of Han and Sofonea (2002), Haslinger and Mäkinen (2003) as well as Hüber *et al.* (2008), it follows that the problem (31)–(32) has a unique solution $(u(t),\lambda(t)) \in K \times \Lambda$ such that $\lambda(t) = \sigma_T(u(t))$.

2.2. Topology optimization problem. Before formulating a structural optimization problem for the quasistatic contact system (31)–(32), let us introduce a set U_{ad} of admissible domains. Denote by $\mathrm{Vol}(\Omega)$ the volume of the domain Ω equal to

$$\mathrm{Vol}(\Omega) = \int_\Omega \mathrm{d}x. \tag{33}$$

The domain Ω is assumed to satisfy the volume constraint of the form

$$\mathrm{Vol}(\Omega) - \mathrm{const}_0 \leq 0, \tag{34}$$

where the constant $\mathrm{const}_0 > 0$ is given. This constant is usually assumed to be the volume of the initial domain Ω and (34) is satisfied (Myśliński, 2008) in the form

$$\mathrm{Vol}(\Omega) = r_{\mathrm{fr}}\,\mathrm{const}_0 \quad \text{with} \quad r_{\mathrm{fr}} \in (0,1). \tag{35}$$

The set U_{ad} of admissible domains has the following form:

$$U_{\mathrm{ad}} = \{\Omega \subset O_l \; : \; \Omega \text{ satisfies (34)}, \; P_D(\Omega) \leq \mathrm{const}_1\}. \tag{36}$$

The perimeter $P_D(\Omega)$ of a domain Ω in D is defined (Sokołowski and Zolesio, 1992, p. 126) by $P_D(\Omega) = \int_\Gamma \mathrm{d}x$. The constant $\mathrm{const}_1 > 0$ is given. The set U_{ad} is assumed to be nonempty.

In order to formulate an optimization problem for the contact system (31)–(32), let us define the shape functional approximating the normal contact stress on the contact boundary. This functional has the form (Myśliński, 2006; 2008):

$$J_\phi(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u(t))\phi_N(x)\,\mathrm{d}s. \tag{37}$$

This cost functional depends on the solution $u = u(t)$ to (31)–(32) in domain Ω for almost all $t \in I$ and on the given auxiliary bounded function $\phi(x) \in M^{st}$. The set M^{st} of auxiliary functions is

$$M^{st} = \{\phi \in H^1(D;\mathbb{R}^2) \; : \phi_i \leq 0 \text{ on } D, \; i = 1,2, \; \| \phi \|_{H^1(D;\mathbb{R}^2)} \leq 1 \}, \tag{38}$$

where the norm $\| \phi \|_{H^1(D;R^2)} = (\sum_{i=1}^2 \| \phi_i \|^2_{H^1(D)})^{1/2}$ (Myśliński, 2006). σ_N and ϕ_N are the normal components of the stress field σ corresponding to a solution u satisfying (31)–(32) and the function ϕ, respectively.

We shall consider the following topology optimization problem: For a given function $\phi \in M^{st}$, find a domain $\Omega^\star \in U_{\rm ad}$ such that

$$J_\phi(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\phi(u(\Omega)). \qquad (39)$$

The aim of the topological optimization problem (39) is to find such a material distribution inside the domain Ω occupied by the elastic body so as to minimize its normal contact stress. For the sake of clarity, let us remark that the paper is confined to considering topology perturbations of the domain Ω only. These topology perturbations consist in nucleation or merging holes or weaker materials inside domain Ω (Amstuz *et al.*, 2008; Eschenauer *et al.*, 1994; Nazarov and Sokołowski, 2003; Novotny *et al.*, 2005; Sokołowski and Żochowski, 1999; 2004) and are performed without any *a priori* assumption about the domain's topology. We do not consider a case of simultaneous boundary perturbations of the domain Ω as in the classical shape optimization (see Sokołowski and Żochowski, 2004) and topology perturbations. These boundary perturbations, consisting in moving the domain boundary in the direction of a suitable velocity field, are performed under the assumption that the initial and final shape domains have the same topology (Garreau *et al.*, 2001; Sokołowski and Żochowski, 2004).

Recall after Chambolle (2003) that the class of domains O_l determined by (1) is endowed with the complementary Hausdorff topology that guarantees the class itself to be compact. The admissibility condition $^\#\Omega^c \leq l$ is crucial to provide the necessary compactness property of $U_{\rm ad}$ (Chambolle, 2003). The existence of an optimal domain $\Omega^\star \in U_{\rm ad}$ to the topology optimization problem (39) follows from the Šverák theorem and arguments provided by Chambolle (2003, Theorem 2).

3. Topological sensitivity analysis

Consider minimization of the the domain functional (39) and topology variations of the domain Ω. Topology variations of geometrical domains are based on the creation of a small hole or a void

$$B(x,\rho) = \{z \in \mathbb{R}^2 \ : \ |x - z| < \rho\} \subset \Omega \qquad (40)$$

with radius ρ, such that $0 < \rho < R$, $R > 0$, given at a point $x \in \Omega$ in the interior of the domain Ω (see Fig. 2). For simplicity it is assumed that $0 \in \Omega$ and we shall consider the case $x = 0$. The closure and the measure of the hole $B(x,\rho)$ are denoted by $\overline{B(x,\rho)}$ and $|\overline{B(x,\rho)}|$, respectively. The domain Ω perturbed by the hole $B(x,\rho)$ is denoted by $\Omega_\rho = \Omega \setminus \overline{B(x,\rho)}$. Moreover, $\Gamma_\rho = \partial B(x,\rho)$ denotes the inner boundary of the domain Ω_ρ. Remark that the hole may also be considered (Allaire *et al.*, 2004) as filled with a weaker material than the material of the domain Ω. This weaker material is characterized by a lower value of the Young modulus. Denote by W_ρ, F_ρ and K_ρ the spaces and the set of kinematically admissible displacements defined by (14)–(16) in domain Ω_ρ rather than Ω. The contact problem (31)–(32) reformulated in domain $\Omega_\rho \times I$ has the following form: Find $(u_\rho, \lambda_\rho) \in W_\rho \times \tilde{\Lambda}$ such that $(u_\rho(t), \lambda_\rho(t)) \in K_\rho \times \Lambda$ satisfies

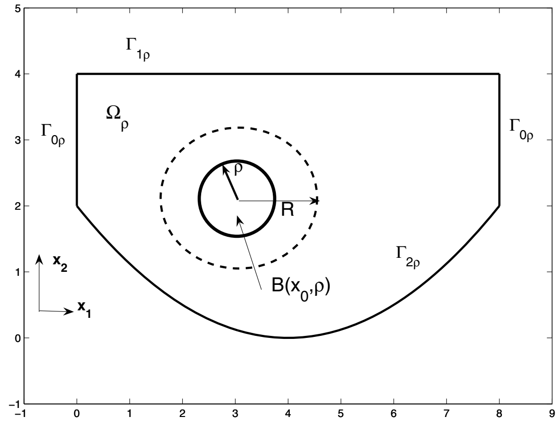

Fig. 2. Perturbed domain Ω_ρ.

$$a(u_\rho, v - \dot{u}_\rho) + \int_{\Gamma_2} \lambda_\rho \cdot (v_T - \dot{u}_{\rho T})\,\mathrm{d}s \geq \int_{\Omega_\rho} f \cdot (v - \dot{u}_\rho)\,\mathrm{d}x + \int_{\Gamma_1} p \cdot (v - \dot{u}_\rho)\,\mathrm{d}s, \quad \forall v \in K_\rho, \qquad (41)$$

$$\int_{\Gamma_2} \lambda \cdot \dot{u}_{\rho T}\,\mathrm{d}s \leq \int_{\Gamma_2} \zeta \cdot \dot{u}_{\rho T}\,\mathrm{d}s, \quad \forall \zeta \in \Lambda, \qquad (42)$$

with $(u_\rho(0), \lambda_\rho(0)) \in K_\rho \times \Lambda$ satisfying (25) and (29), respectively, in the domain Ω_ρ, rather than Ω. The restriction of the test function φ to Ω_ρ is also denoted by φ. The displacement u_ρ satisfies the homogeneous Neumann boundary condition on the inner boundary Γ_ρ. The topology variations of geometrical domains may be defined as functions of a small parameter ρ. Recall after Garreau *et al.* (2001) as well as Sokołowski and Żochowski (1999; 2004; 2005) the definition of the topological derivative of the domain functional.

Definition 1. The *topological derivative* $TJ_\phi(\Omega, x)$ of the domain functional $J_\phi(\Omega)$ at $\Omega \subset \mathbb{R}^2$ is a function depending on a centre x of the small hole and is defined by

$$TJ_\phi(\Omega, x) = \lim_{\rho \to 0^+} \frac{J_\phi(\Omega \setminus \overline{B(x,\rho)}) - J_\phi(\Omega)}{|\overline{B(x,\rho)}|}. \qquad (43)$$

Since in the paper we confine ourselves to considering the creation of small holes of a circular shape only, in this case the measure of the hole equals its volume, i.e., $|\overline{B(x,\rho)}| = \pi\rho^2$.

Using the methodology of Sokołowski and Żochowski (1999; 2004) as well as the results of differentiability of solutions to variational inequalities (Sokołowski and Zolesio, 1992), let us calculate directly from the definition (43) the formulae of the topological derivative $TJ_\phi(\Omega; x_0)$ of the cost functional (37) at a point $x_0 \in \Omega$.

Write $J_\phi(\Omega_\rho) = J_\phi(\rho)$. Assuming that the following expansion holds for $\Omega \subset \mathbb{R}^2$:

$$J_\phi(\rho) = J_\phi(0) + \frac{\rho^2}{2} J''_\phi(0^+) + o(\rho^2), \tag{44}$$

where $J''_\phi(\rho)$ denotes the second order derivative of $J_\phi(\rho)$ with respect to ρ, the topological derivative $TJ_\phi(\Omega, x)$ equals (Sokołowski and Żochowski, 2004)

$$TJ_\phi(\Omega, x) = \frac{1}{2\pi} J''_\phi(0^+). \tag{45}$$

3.1. Topological derivative form.

Lemma 1. *Let Assumptions (i)–(iii) of Theorem 1 be satisfied. The topological derivative $TJ_\phi(\Omega, x_0)$ of the domain functional (37) at a point $x_0 \in \Omega$ for almost all $t \in I$ has the form*

$$\begin{aligned} &TJ_\phi(\Omega, x_0) \\ &\quad = -[f(t)\cdot(\phi + r(t)) + \frac{1}{E}(a_{u(t)} a_{r(t)+\phi} \\ &\qquad + 2b_{u(t)} b_{r(t)+\phi} \cos 2\delta)]_{|x=x_0}, \end{aligned} \tag{46}$$

where E denotes the Young modulus and

$$a_{u(t)} \stackrel{\text{def}}{=} \sigma_I(u(t)) + \sigma_{II}(u(t)), \tag{47}$$

$$b_{u(t)} \stackrel{\text{def}}{=} \sigma_I(u(t)) - \sigma_{II}(u(t)). \tag{48}$$

$\sigma_I(u(t))$ and $\sigma_{II}(u(t))$ denote the principal stresses corresponding to the displacement $u(t) \in K$ satisfying the system (41)–(42) and stress tensor elements $\sigma_{ij}(u(t))$, $i, j = 1, 2$. They are equal to

$$\sigma_I(u(t)) = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{(\frac{\sigma_{11} - \sigma_{22}}{2})^2 + \sigma_{12}^2}, \tag{49}$$

$$\sigma_{II}(u(t)) = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{(\frac{\sigma_{11} - \sigma_{22}}{2})^2 + \sigma_{12}^2}. \tag{50}$$

Here δ is the angle between principal stresses directions determined by

$$\tan 2\delta = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}. \tag{51}$$

The adjoint state $(r_\rho, q_\rho) \in W_\rho \times \tilde{\Lambda}$ satisfies in the domain Ω_ρ the following system: Find $(r_\rho(t), q_\rho(t)) \in K_{\rho 1} \times \Lambda_1$ such that

$$a(r_\rho(t) + \phi, \varphi) + \int_{\Gamma_2} q_\rho(t) \cdot \dot{\varphi}_T \, \mathrm{d}s = 0, \quad \forall \varphi \in W_\rho \ \text{s.t.}\ \varphi(t) \in K_{\rho 1}, \tag{52}$$

$$\int_{\Gamma_2} (\phi_T + r_{\rho T}(t)) \cdot \zeta \, \mathrm{d}s = 0, \quad \forall \zeta \in \Lambda_1, \tag{53}$$

with $(r_\rho(\mathcal{T}), q_\rho(\mathcal{T})) \in K_{\rho 1} \times \Lambda_1$. Moreover, we have $r_\rho(x,t)_{|\rho=0} = r(x_0, t)$. The sets $K_{\rho 1}$ and Λ_1 are given by

$$K_{\rho 1} = \{\xi \in F_\rho \ : \ \xi_N = 0 \ on\ A^{st}\}, \tag{54}$$

$$\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \ on\ B_1 \cup B_2 \cup B_1^+ \cup B_2^+\}, \tag{55}$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N = 0\}$. Moreover, $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$.

Note that using the arguments similar to those in the proof of Theorem 1 we can show the existence of the solution to the system (52)–(53). This variational system has the following operator form:

$$\sigma_{ij}(r_\rho + \phi)_{,j} = 0 \quad \text{in} \quad \Omega_\rho \times I, \quad i, j = 1, 2, \tag{56}$$

$$\sigma_N(r_\rho + \phi) = 0 \quad \text{on} \quad \Gamma_1 \times I, \tag{57}$$

$$\phi_T + r_{\rho T} = 0, \quad \sigma_T(r_\rho + \phi) - \dot{q}_\rho = 0 \ \text{ on } \Gamma_2 \times I. \tag{58}$$

Proof. It follows the ideas of shape sensitivity analysis (Sokołowski and Żochowski, 2004) and is based on direct application of (43), a shape derivative approach (Sokołowski and Żochowski, 2004) and asymptotic expansion methods (Sokołowski and Żochowski, 1999). Here we provide the main steps only.

Consider a time discretization of the state system (41)–(42). For a sequence of increasing integers $m \geq 1$ let us define the stepsize $\Delta t = \mathcal{T}/m$ and the grid points $t_k = k \cdot \Delta t$, $k = 0, \ldots, m$. Write $u^k = u(x, t_k)$, $u^k = u^k(x)$, $f^k = f(x, t_k)$, $p^k = p(x, t_k)$, $\lambda^k = \lambda(x, t_k)$. Using an implicit scheme as well as approximating the time derivative $\dot{u}_\rho^{k+1}$ by the finite difference

$$\dot{u}_\rho^{k+1} \approx \frac{u_\rho^{k+1} - u_\rho^k}{\Delta t},$$

multiplying (41) by Δt and setting $v := u_\rho^k + \Delta t v$, we obtain the discretized system of variational inequalities (41)–(42): Find $(u_\rho^{k+1}, \lambda_\rho^{k+1}) \in K_\rho \times \Lambda$, $k = 0, \ldots, m$, satisfying

$$\begin{aligned} &a(u_\rho^{k+1}, v - u_\rho^{k+1}) + \int_{\Gamma_2} \lambda_\rho^{k+1} \cdot (v_T - u_{\rho T}^{k+1}) \, \mathrm{d}s \\ &\quad \geq \int_{\Omega_\rho} f^{k+1} \cdot (v - u_\rho^{k+1}) \, \mathrm{d}x \\ &\qquad + \int_{\Gamma_1} p^{k+1} \cdot (v - u_\rho^{k+1}) \, \mathrm{d}s, \qquad \forall v \in K_\rho, \end{aligned} \tag{59}$$

$$\int_{\Gamma_2} \lambda^{k+1} \cdot (u_{\rho T}^{k+1} - u_{\rho T}^{k})\,\mathrm{d}s \le \int_{\Gamma_2} \zeta \cdot (u_{\rho T}^{k+1} - u_{\rho T}^{k})\,\mathrm{d}s, \quad \forall \zeta \in \Lambda. \tag{60}$$

The problem (59)–(60) is nothing but a weak formulation of the static contact problem with a given friction (Duvaut and Lions, 1972; Eck *et al.*, 2005; Han and Sofonea, 2002; Myśliński, 2006). Here u_ρ^{k+1} satisfies the system (7)–(11) in domain Ω_ρ and the following friction condition on Γ_2 at time $t = t_{k+1}$:

$$|\sigma_T(u_\rho^{k+1})| < 1 \quad \Rightarrow \quad u_{\rho T}^{k+1} = u_{\rho T}^{k}, \tag{61}$$

$$|\sigma_T(u_\rho^{k+1})| = 1 \Rightarrow \quad \exists \lambda \ge 0 \ \text{ s.t. } \ (u_{\rho T}^{k+1} - u_{\rho T}^{k}) = -\lambda \sigma_T(u_\rho^{k+1}). \tag{62}$$

From the Green formula (Duvaut and Lions, 1972; Han and Sofonea, 2002) it follows that the domain functional (37) is associated with the solutions to this state system through the following relation:

$$\begin{aligned} J_\phi(\rho) &\stackrel{\text{def}}{=} J_\phi(u(\Omega_\rho)) = \int_{\Gamma_2} \sigma_{\rho N} \phi_N\,\mathrm{d}s \\ &= \int_{\Omega_\rho} \sigma_{ij}(u_\rho) \varepsilon_{\tilde{k}l}(\phi)\,\mathrm{d}x - \int_{\Omega_\rho} f_i \phi_i\,\mathrm{d}x \\ &\quad - \int_{\Gamma_1} p_i \phi_i\,\mathrm{d}s + \int_{\Gamma_2} \lambda_\rho \cdot \phi_T\,\mathrm{d}s, \end{aligned} \tag{63}$$

for $i, j, \tilde{k}, l = 1, 2$. At $t = t_{k+1}$, this functional takes the form

$$\begin{aligned} J_\phi(u_\rho^{k+1}) &= \int_{\Omega_\rho} \sigma_{ij}(u_\rho^{k+1}) \varepsilon_{\tilde{k}l}(\phi)\mathrm{d}x - \int_{\Omega_\rho} f_i^{k+1} \phi_i \mathrm{d}x \\ &\quad - \int_{\Gamma_1} p_i^{k+1} \phi_i \mathrm{d}s + \int_{\Gamma_2} \lambda_\rho^{k+1} \cdot \phi_T \mathrm{d}s, \\ &\quad i, j, \tilde{k}, l = 1, 2. \end{aligned} \tag{64}$$

Using the formulae of domain functional derivatives with respect to the ball radius ρ (Sokołowski and Żochowski, 2004, (27)–(28), p. 63) considered to be a particular case of shape derivatives and differentiating (64) with respect to ρ, we get the shape derivative of the domain functional (37) at time $t = t_{k+1}$. Assuming that neither the surface traction p nor the boundaries Γ_1 and Γ_2 are dependent on ρ, this derivative has the form (Myśliński, 2006)

$$\begin{aligned} J'_\phi(u_\rho^{k+1}) &= \int_{\Omega_\rho} \sigma_{ij}(u_\rho^{k+1\prime}) \varepsilon_{kl}(\phi)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_\rho} \sigma_{ij}(u_\rho^{k+1}) \varepsilon_{kl}(\phi)\,\mathrm{d}s + \int_{\Gamma_\rho} f_i^{k+1} \phi_i\,\mathrm{d}s \\ &\quad + \int_{\Gamma_2} (\lambda_\rho^{k+1\prime} \cdot \phi_T + \frac{1}{\rho} \lambda_\rho^{k+1} \cdot \phi_T)\,\mathrm{d}s, \\ &\quad i, j, \tilde{k}, l = 1, 2. \end{aligned} \tag{65}$$

The cost functional derivative (65) depends on the shape derivative $(u_\rho^{k+1\prime}, \lambda_\rho^{k+1\prime})$ of the solution $(u_\rho^{k+1}, \lambda_\rho^{k+1}) \in K_\rho \times \Lambda$ to the contact problem (59)–(60). Using the results concerning the regularity of solutions to the state system (41)–(42) (Myśliński, 2006) and the differentiability of solutions to variational inequalities (Sokołowski and Zolesio, 1992, Theorem 4.3.3, p. 213), one can show that the shape derivative $(u_\rho^{k+1\prime}, \lambda_\rho^{k+1\prime}) \in K_{\rho 1} \times \Lambda_1$ of the solution $(u_\rho^{k+1}, \lambda_\rho^{k+1}) \in K_\rho \times \Lambda$ to the contact problem (59)–(60) satisfies the following system:

$$\begin{aligned} &\int_{\Omega_\rho} \sigma_{ij}(u_\rho^{k+1\prime}) e_{kl}(\varphi)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_\rho} (\sigma_{ij}(u_\rho^{k+1}) e_{kl}(\varphi) - f_i^{k+1} \varphi_i)\,\mathrm{d}s \\ &\quad + \int_{\Gamma_2} (\lambda_\rho^{k+1\prime} \varphi_T + \frac{1}{\rho} \lambda_\rho^{k+1} \varphi_T) \mathrm{d}s \ge 0, \qquad \forall \varphi \in K_{\rho 1}, \end{aligned} \tag{66}$$

$$\int_{\Gamma_2} u_{\rho T}^{k+1\prime} \cdot \zeta\,\mathrm{d}s = 0, \quad \forall \zeta \in \Lambda_1. \tag{67}$$

Let us define an adjoint system at time $t = t_{k+1}$. The adjoint state $(r_\rho^{k+1}, q_\rho^{k+1}) \in K_{\rho 1} \times \Lambda_1$ satisfies in domain Ω_ρ the following system:

$$a(r_\rho^{k+1} + \phi, \varphi^{k+1}) + \int_{\Gamma_2} q_\rho^{k+1} \cdot \dot{\varphi}_T^{k+1}\,\mathrm{d}s = 0, \quad \forall \varphi^{k+1} \in W_\rho \cap K_{\rho 1}, \tag{68}$$

$$\int_{\Gamma_2} (\phi_T + r_{\rho T}^{k+1}) \cdot \eta^{k+1}\,\mathrm{d}s = 0, \quad \forall \eta^{k+1} \in \Lambda_1. \tag{69}$$

Setting $\varphi = r_\rho^{k+1}$ and $\zeta = q_\rho^{k+1}$ in the shape derivative system (66)–(67), as well as $\varphi^{k+1} = u_\rho^{k+1\prime}$ and $\eta^{k+1} = \lambda_\rho^{k+1\prime}$ in the adjoint system (68)–(69), the domain functional derivative (65) takes the form

$$\begin{aligned} J'_\phi(\rho) = -\int_{\Gamma_\rho} &[\sigma_{ij}(u_\rho^{k+1}) e_{kl}(\phi + r_\rho^{k+1}) \\ &- f^{k+1} \cdot (\phi + r_\rho^{k+1})]\,\mathrm{d}s. \end{aligned} \tag{70}$$

Consider the shape derivative (70) for $\rho \to 0^+$. To calculate the derivative (46), we use asymptotic expansions of solutions to elasticity systems in $\mathbb{R}^2$ in a polar coordinates system (Sokołowski and Żochowski, 2004). Consider the coordinate system aligned with the direction of principal stresses σ_I and σ_{II}. We introduce the polar coordinate system (r, θ) with the coordinate axis still denoted by (r, θ).

The displacement $u_\rho^{k+1} = (u_{\rho r}^{k+1}, u_{\rho\theta}^{k+1})$ is a function of r and θ in polar coordinates. The following asymptotic expansions of u_ρ^{k+1} in the ring $\rho \le r \le 2\rho$ hold (Sokołowski and Żochowski, 2004):

$$\begin{aligned} u_{\rho r}^{k+1} &= u_r^{k+1}(0) + \frac{a_{u^{k+1}}}{8Gr}\Big[(\kappa^2-1)r^2 + 2\rho^2\Big] \\ &\quad + \frac{b_{u^{k+1}}}{4Gr}\Big[(\kappa+1)\rho^2 + r^2 - \frac{\rho^4}{r^2}\Big] \\ &\quad \times \cos(2\theta) + O(\rho^{2-\varepsilon}), \end{aligned} \tag{71}$$

$$\begin{aligned} u_{\rho\theta}^{k+1} &= u_\theta^{k+1}(0) - \frac{b_{u^{k+1}}}{4Gr}\Big[(\kappa-1)\rho^2 \\ &\quad + r^2 + \frac{\rho^4}{r^2}\Big]\sin(2\theta) + O(\rho^{2-\varepsilon}), \end{aligned} \tag{72}$$

where for $\varepsilon > 0$ the reminder $O(\rho^{2-\varepsilon}) \to 0$ as $\rho \to 0^+$, as well as

$$G = \frac{E}{2(1+\nu)}, \qquad \kappa = \frac{(3-\nu)}{(1+\nu)}, \tag{73}$$

and

$$\begin{aligned} \lim_{r\to 0} u_r^{k+1}(r,\theta) &= u_r^{k+1}(0), \\ \lim_{r\to 0} u_\theta^{k+1}(r,\theta) &= u_\theta^{k+1}(0). \end{aligned} \tag{74}$$

Using the asymptotic expansions (71)–(72) as well as the strain expressions and Hook's law in the polar coordinate system (Sokołowski and Żochowski, 2004, p. 92–93), the asymptotic expansions for elements of a stress tensor have the following form (Sokołowski and Żochowski, 2004):

$$\begin{aligned} &\sigma_{rr}(u_\rho^{k+1}) \\ &\quad = \frac{1}{2}\Big[a_{u^{k+1}}(1 - \frac{\rho^2}{r^2}) + b_{u^{k+1}}(1 - 4\frac{\rho^2}{r^2} \\ &\qquad + 3\frac{\rho^4}{r^4}\cos(2\theta))\Big] + O(\rho^{1-\epsilon}), \end{aligned} \tag{75}$$

$$\begin{aligned} &\sigma_{\theta\theta}(u_\rho^{k+1}) \\ &= \frac{1}{2}\Big[a_{u^{k+1}}(1 + \frac{\rho^2}{r^2}) - b_{u^{k+1}}(1 + 3\frac{\rho^4}{r^4}\cos(2\theta))\Big] \\ &\quad + O(\rho^{1-\epsilon}), \end{aligned} \tag{76}$$

$$\begin{aligned} &\sigma_{r\theta}(u_\rho^{k+1}) \\ &= -\frac{1}{2}\Big[b_{u^{k+1}}(1 + 2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4})\sin(2\theta)\Big] + O(\rho^{1-\epsilon}). \end{aligned} \tag{77}$$

The free edge condition on the boundary Γ_ρ of the hole results in $\sigma_{rr}(u_\rho^{k+1}) = \sigma_{r\theta}(u_\rho^{k+1}) = 0$. Using it, along with the relation between the stress tensor components in the Cartesian and polar coordinate systems, the derivative (70) takes the form

$$\begin{aligned} J'_\phi(u_\rho^{k+1}) = -\int_{\Gamma_\rho} &\Big[\frac{1}{E}\sigma_{\theta\theta}(r_\rho^{k+1} + \phi)\sigma_{\theta\theta}(u_\rho^{k+1}) \\ &+ f^{k+1}\cdot(\phi + r_\rho^{k+1})\Big]\,\mathrm{d}s. \end{aligned} \tag{78}$$

From (20), (38) and the asymptotic expansions (75)–(77), it follows that all integrands in (78) are bounded. Therefore, we have that

$$\lim_{\rho\to 0^+} J'_\phi(u_\rho^{k+1}) = 0. \tag{79}$$

Using once more the formulae for the derivatives of the domain functionals (Sokołowski and Zolesio, 1992) and differentiating the functional (78) with respect to ρ, we obtain

$$J''_\phi(\rho) = J_1(\rho) + J_2(\rho) + J_3(\rho), \tag{80}$$

where

$$\begin{aligned} J_1(\rho) = \int_{\Gamma_\rho} \frac{\partial}{\partial n}\Big[&\frac{1}{E}\sigma_{\theta\theta}(r_\rho^{k+1} + \phi)\sigma_{\theta\theta}(u_\rho^{k+1}) \\ &+ f^{k+1}\cdot(\phi + r_\rho^{k+1})\Big]\,\mathrm{d}s, \end{aligned} \tag{81}$$

$$\begin{aligned} J_2(\rho) = -\int_{\Gamma_\rho} \Big[&\frac{1}{E}(\sigma_{\theta\theta}(r_\rho^{k+1} + \phi)\sigma_{\theta\theta}(u_\rho^{k+1}))' \\ &+ f^{k+1}\cdot(\phi + r_\rho^{k+1})'\Big]\,\mathrm{d}s, \end{aligned} \tag{82}$$

$$\begin{aligned} J_3(\rho) = -\frac{1}{\rho}\int_{\Gamma_\rho} \Big[&\frac{1}{E}\sigma_{\theta\theta}(r_\rho^{k+1} + \phi)\sigma_{\theta\theta}(u_\rho^{k+1}) \\ &+ f^{k+1}\cdot(\phi + r_\rho^{k+1})\Big]\,\mathrm{d}s. \end{aligned} \tag{83}$$

Recall after Sokołowski and Żochowski (2004) that on Γ_ρ

$$\frac{\partial}{\partial n}(\cdot) = -\frac{\partial}{\partial r}(\cdot).$$

Using the asymptotic expansion (76), we obtain on Γ_ρ the derivatives of $\sigma_{\theta\theta}$,

$$\begin{aligned} \frac{\partial \sigma_{\theta\theta}(u_\rho^{k+1})}{\partial n} &= a_{u^{k+1}}\frac{\rho^2}{r^3} - 6b_{u^{k+1}}\frac{\rho^4}{r^5}\cos(2\theta) \\ &\overset{r=\rho}{=} a_{u^{k+1}}\frac{1}{\rho} - 6b_{u^{k+1}}\frac{1}{\rho}\cos(2\theta), \end{aligned} \tag{84}$$

$$\begin{aligned} &\frac{\partial \sigma_{\theta\theta}(u_\rho^{k+1})}{\partial \rho} \\ &\quad = a_{u^{k+1}}\frac{\rho}{r^2} - 6b_{u^{k+1}}\frac{\rho^3}{r^4}\cos(2\theta) + O(\rho^{-\varepsilon}) \\ &\quad \overset{r=\rho}{=} a_{u^{k+1}}\frac{1}{\rho} - 6b_{u^{k+1}}\frac{1}{\rho}\cos(2\theta) + O(\rho^{-\varepsilon}), \end{aligned} \tag{85}$$

From (84)–(85) we have that singular terms cancel out:

$$\frac{\partial \sigma_{\theta\theta}(u_\rho^{k+1})}{\partial n} - \frac{\partial \sigma_{\theta\theta}(u_\rho^{k+1})}{\partial \rho} = O(\rho^{-\varepsilon}). \tag{86}$$

Using (76) as well as (86) we obtain that, as $\rho \to 0^+$,

$$J_1(\rho) + J_2(\rho) \to 0. \tag{87}$$

Consider the last term $J_3(\rho)$ of the functional (80). The stress component $\sigma_{\theta\theta}(r_\rho^{adt}+\phi)$ can be expressed in the frame of principal stress directions for the displacement field u_ρ^{k+1}, i.e.,

$$\sigma_{\theta\theta}(r_\rho^{k+1}+\phi) = \frac{1}{2}a_{r_\rho^{k+1}+\phi}(1+\frac{\rho^2}{r^2}) - \frac{1}{2}b_{r_\rho^{k+1}+\phi}(1+3\frac{\rho^4}{r^4})\cos 2(\theta-\delta)) + O(\rho^{1-\varepsilon}) \overset{r=\rho}{=} a_{r_\rho^{k+1}+\phi} - 2b_{r_\rho^{k+1}+\phi}\cos 2(\theta-\delta) + O(\rho^{1-\varepsilon}). \tag{88}$$

Using (88) and transforming the integral over Γ_ρ into the integral on interval $[0,2\pi]$, we obtain

$$\lim_{\rho\to 0^+}\int_0^{2\pi}\sigma_{\theta\theta}(u_\rho^{k+1})\sigma_{\theta\theta}(r_\rho^{k+1}+\phi)d\theta = 2\pi\rho[a_{u^{k+1}}a_{r^{k+1}+\phi} + 2b_{u^{k+1}}b_{r^{k+1}+\phi}\cos(2\delta)]. \tag{89}$$

Using (83) and (89) along with the properties of integrals along the circle boundary Γ_ρ and employing the formula (45), we get the topological derivative (46) with u^{k+1}.

Consider the passage to the limit as $k\to\infty$ (i.e., as $\Delta t\to 0$) in this formula. Let $\{u_m\}$ and $\{\tilde{u}_m\}$ denote the sequences of functions defined on $[0,T]$ by $u_m(t) = u^{k+1}$ and $\tilde{u}_m(t) = u^k + (t-t_k)(u^{k+1}-u^k)/\Delta t$ for $t\in(t_k,t_{k+1})$ with $t_k = k\triangle t$, $k=0,1,\ldots,m-1$, and $u_m(0)=\tilde{u}_m(0)=u_0$. From the work of Rocca and Cocu (2001, Lemmas 14–16, Theorem 17) it follows that there exist two subsequences $\{u'_m\}\subset\{u_m\}$ and $\{\tilde{u}'_m\}\subset\{\tilde{u}_m\}$, denoted further by $\{u_m\}$ and $\{\tilde{u}\}$, and an element $u\in H^1(I;F)$ satisfying the system (26) such that

$$u_m(t) \rightharpoonup u(t) \ \text{weakly in}\ F, \tag{90}$$

$$\tilde{u}_m(t) \rightharpoonup u(t) \ \text{weakly in}\ H^1(I;F). \tag{91}$$

Using the same arguments, we obtain

$$r_m(t) \rightharpoonup r(t) \ \text{weakly in}\ F, \tag{92}$$

$$\tilde{r}_m(t) \rightharpoonup r(t) \ \text{weakly in}\ H^1(I;F). \tag{93}$$

Using (90) as well as (2) we obtain that the sequence of stress tensor components for $i,j=1,2$

$$\sigma_{ij}(u_m(t)) \rightharpoonup \sigma_{ij}(u(t)) \ \text{weakly in}\ H^1(I;\mathcal{H}). \tag{94}$$

From the results of Rocca and Cocu (2001, Lemma 15) as well as (18)–(19) we get the boundedness of the sequence of the stress fields $\sigma(u_m(t))$. Using (49)–(50) as well as (47)–(48) and denoting by α either $\alpha=u$ or $\alpha=r$, we obtain

$$\sigma_I(u_m(t)) \rightharpoonup \sigma_I(u(t)) \ \text{weakly in}\ H^1(I;\mathcal{H}), \tag{95}$$

$$\sigma_{II}(u_m(t)) \rightharpoonup \sigma_{II}(u(t)) \ \text{weakly in}\ H^1(I;\mathcal{H}), \tag{96}$$

$$a_{\alpha_m(t)} \rightharpoonup a_{\alpha(t)} \ \text{weakly in}\ H^1(I;\mathcal{H}), \tag{97}$$

$$b_{\alpha_m(t)} \rightharpoonup b_{\alpha(t)} \ \text{weakly in}\ H^1(I;\mathcal{H}). \tag{98}$$

Assuming that $f(t_m)$ converges strongly in $L^2(I;L^2(\Omega;R^2))$ to $f(t)$ and using (93) and (95)–(98), we obtain that the sequence of the topological derivatives (46) for $u(t_m)$ converges to the topological derivative (46) for $u(t)$ satisfying (31)–(32) as $\Delta t\to 0$ ($m\to\infty$). ■

3.2. Necessary optimality condition. Using the topological derivative (46), the following necessary optimality condition for the problem (39) can be formulated.

Lemma 2. *Let $\Omega^\star\in U_{\rm ad}$ be an optimal solution to the problem (39). Then there exist Lagrange multipliers $\mu_1\in\mathbb{R}$, $\mu_1\geq 0$, associated with the volume constraint and $\mu_2\in\mathbb{R}$, $\mu_2\geq 0$, associated with the finite perimeter constraint such that for all $x\in\Omega^\star$ and for all topology perturbations $\delta\Omega\in U_{\rm ad}$ of the domain $\Omega^\star\in U_{\rm ad}$ given by (40) such that $\Omega^\star\cup\delta\Omega\in U_{\rm ad}$, at any optimal solution $\Omega^\star\in U_{\rm ad}$ to the topology optimization problem (39), the following conditions are satisfied for almost all $t\in I$:*

$$TJ_\phi(u(\Omega^\star);x) + \mu_1 + \mu_2\,{\rm d}P_D(\Omega^\star;x) \geq 0, \tag{99}$$

$$(\mu_1^{\sim}-\mu_1)(\int_{\Omega^\star}{\rm d}x - {\rm const}_0) \leq 0, \quad \forall\mu_1^{\sim}\in\mathbb{R},\ \mu_1^{\sim}\geq 0, \tag{100}$$

$$(\mu_2^{\sim}-\mu_2)(P_D(\Omega^\star) - {\rm const}_1) \leq 0, \quad \forall\mu_2^{\sim}\in\mathbb{R},\ \mu_2^{\sim}\geq 0, \tag{101}$$

where $u(\Omega^\star) = u(x,t)$ denotes the solution to the state system (31)–(32) in the domain $\Omega^\star$, the topological derivative $TJ_\phi(u(\Omega^\star);x)$ is given by (46) and ${\rm d}P_D(\Omega^\star;x)$ denotes the topological derivative of the finite perimeter functional $P_D(\Omega^\star)$ equal to

$$dP_D(\Omega^\star;x) = 4\pi, \tag{102}$$

(see Fulmański et al., 2007; Sokołowski and Żochowski, 2004). The given constants ${\rm const}_0>0$ and ${\rm const}_1>0$ are the same as in (36).

The method to prove Lemma 2 is standard (see Duvaut and Lions, 1972; Haslinger and Mäkinen, 2003; Myśliński, 2006). Note that Lemma 2 deals with topology perturbations of domain Ω in the form of circular-shaped holes (40) only.

4. Conclusions

A topology optimization problem for the quasistatic contact phenomenon with prescribed friction has been considered in the paper. The topology derivative of the shape functional has been calculated and the necessary optimality condition formulated. The calculated derivative will be used in numerical topology optimization of the rolling contact problem where one of the contacting surfaces is covered with the functionally graded material (see Fig. 3).

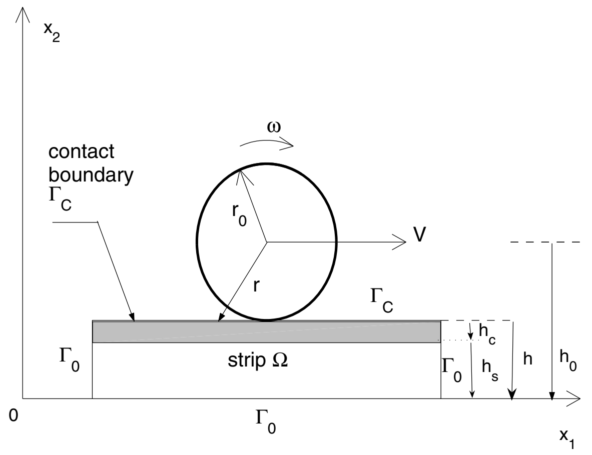

Fig. 3. Quasistatic wheel-rail rolling contact problem.

This problem is described by the quasistatic elliptic variational inequality (Chudzikiewicz and Myśliński, 2009). The topology optimization approach is useful in reducing the normal contact stress, which is the main factor generating rolling contact fatigue and noise during the movement of the body in contact with the foundation. Examples of numerical solutions of topological optimization problems for static contact problems can be found in the work of Myśliński (2010).

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Andrzej Myśliński received the M.Sc. degree in control engineering in 1978 from the Warsaw University of Technology as well as the Ph.D. and D.Sc. degrees in control engineering in 1987 and 2007, respectively, from the Systems Research Institute of the Polish Academy of Sciences, where he has been working at the Systems Research Institute since 1981, currently as an associate professor. His main research interests include shape and topology optimization of nonlinear distributed parameter systems, image processing based on the variational approach, numerical methods of optimization and the modelling of unilateral boundary value problems.

Received: 21 March 2011
Revised: 8 September 2011