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H_∞ CONTROL OF DISCRETE–TIME LINEAR SYSTEMS CONSTRAINED IN STATE BY EQUALITY CONSTRAINTS

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In this paper, stabilizing problems in control design are addressed for linear discrete-time systems, reflecting equality constraints tying together some state variables. Based on an enhanced representation of the bounded real lemma for discrete-time systems, the existence of a state feedback control for such conditioned stabilization is proven, and an LMI-based design procedure is provided. The control law gain computation method used circumvents generally an ill-conditioned singular design task. The principle, when compared with previously published results, indicates that the proposed method outperforms the existing approaches, guarantees feasibility, and improves the steady-state accuracy of the control. Furthermore, better performance is achieved with essentially reduced design effort. The approach is illustrated on simulation examples, where the validity of the proposed method is demonstrated using one state equality constraint.

Keywords: equality constraints, discrete-time systems, linear matrix inequality, state feedback, control algorithms, quadratic stability, singular systems.

1. Introduction

In the last years many results have spurred interest in the problem of control determination for systems with constraints. In the typical case (Benzaouia and Gurgat, 1988; Castelan and Hennet, 1992), where a system state reflects certain physical entities, this class of constraints often appears because of physical limits, and these constraints usually keep the system state in the region of technological conditions. Subsequently, this problem can be formulated using a technique dealing with system state constraints directly, where it can be coped efficiently using modified linear system techniques (Ko and Bitmead, 2007b). Notably, a special form of constrained problems was defined where the system state variables satisfy constraints (Hahn, 1992; Kaczorek, 2002; Tarbouriech and Castelan, 1995), interpreted as descriptor systems. Because, generally, a system with state equality constraints does not satisfy the conditions under which the results of descriptor systems can be applicable, this approach is very limited. If the design task is interpreted as a singular problem (Krokavec and Filasová, 2008a), constrained methods can be developed to design the controller parameters.

In principle, it is possible and ever easy to design a controller that stabilizes the systems and simultaneously forces closed-loop systems to satisfy constraints (Oloomi and Shafai, 1997; Yu *et al.*, 1996). Following the ideas of Linear Quadratic (LQ) control (Dórea and Milani, 1995; Petersen, 2006; Xue *et al.*, 2006), one direct connected technique, using the equality constraint formulation for discrete-time Multi-Input/Multi-Output (MIMO) systems, was introduced by Ko and Bitmead (2007a) and extensively used in reconfigurable control design (Krokavec and Filasová, 2008b; 2009). Based on the eigenstructure assignment principle, a slight modification of this technique, with application in state constrained control system design, was presented by Filasová and Krokavec (2010).

A number of problems that arise in state feedback control can be reduced to a handful of standard convex and quasi-convex problems that involve matrix inequalities. It is known that optimal solutions can be computed by using the interior point method (Nesterov and Nemirovsky, 1994), which converges in polynomial time with respect to the problem size, and efficient interior point algorithms have recently been developed, while a further development of algorithms for these standard problems is an area of active research. In such an approach, stability conditions may be expressed in terms of Linear Matrix Inequalities (LMIs), which have attracted a notable practical interest due to the existence of numerical solvers (Gahinet *et al.*, 1995; Peaucelle *et al.*, 2002). Some progress review in this field can be found in the works of Boyd *et al.* (1994), Herrmann *et al.* (2007), Skelton *et al.* (1998), and the references therein.

This paper aims at providing controller design conditions for discrete-time systems where the closed-loop state variables are allowed to satisfy the prescribed rations. Based on a Lyapunov function being quadratic in the state and linear in the parameters, and extended to a given quadratic performance using an enhanced LMI representation of the Bounded Real Lemma (BRL), a state control is established in the presence of state equality constraints. Such a restriction does not lead to more conservative results, and design conditions are simple to be established as a set of LMIs which can be solved numerically with the help of an LMI solver. Motivated by the underlying ideas (Filasová and Krokavec, 2010; Krokavec and Filasová, 2008; de Oliveira *et al.*, 1999; Wu and Duan, 2006; Xie, 2010), a simple technique for the enhanced BRL representation is obtained, and new criteria to circumvent an ill-conditioned singular task concerning discrete-time systems control design with state equality constraints are given. Due to the introduction of an enhanced LMI representation of the BRL, which exhibits a kind of decoupling between the Lyapunov matrix and the system matrices, the design task is now well conditioned. These conditions still impose some common matrices to obtain control that assures quadratic stability for time-invariant discrete control under defined state equality constraints.

The paper is organized as follows. Starting with problem formulation presented in Section 2, in Section 3 basic preliminaries are given, and an adapted version of a discrete BRL, referred to as the enhanced BRL form, is introduced. These results are used in Section 4 to derive a new convex formulation of stability conditions considering closed-loop state equality constraints. The proposed approach gives a well-conditioned LMI, and leads to a feasible solution on given singular task. Subsequently, in Section 5 one numerical example is presented to illustrate basic properties of the presented method. Section 6 is finally devoted to a brief overview of the properties of the method and to demonstrating the accepted conservatism.

Throughout the paper, the following notation is used: $\boldsymbol{x}^T$ and $\boldsymbol{X}^T$ denote the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, for a square matrix $\boldsymbol{X} > 0$ (respectively $\boldsymbol{X} < 0$) means that $\boldsymbol{X}$ is a symmetric positive definite matrix (respectively a negative definite matrix), the symbol $\boldsymbol{I}_n$ represents the n-th order unit matrix, $\boldsymbol{X}^{\ominus 1}$ denotes the Moore–Penrose pseudoinverse of $\boldsymbol{X}$, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{n\times r}$ the set of all $n \times r$ real matrices, and $\|\cdot\|$ represents the Euclidean norm for vectors and the spectral norm for matrices.

2. Problem formulation

Through this paper the task is concerned with design of a state feedback constrained in state variables, and controlling discrete-time linear dynamic systems given by the set of state equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i), \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{D}\boldsymbol{u}(i), \tag{2}$$

where $\boldsymbol{q}(i) \in \mathbb{R}^n$, $\boldsymbol{u}(i) \in \mathbb{R}^r$, and $\boldsymbol{y}(i) \in \mathbb{R}^m$ are vectors of the state, input and output variables, respectively, nominal system matrices $\boldsymbol{F} \in \mathbb{R}^{n\times n}$, $\boldsymbol{G} \in \mathbb{R}^{n\times r}$, $\boldsymbol{C} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{D} \in \mathbb{R}^{m\times r}$ are real matrices, and $i \in \mathbb{Z}_+$.

In practice (Cakmakci and Ulsoy, 2009; Debiane *et al.*, 2004), ratio control can be used to maintain the relationship between two state variables, defined as

$$\frac{q_h(i+1)}{q_k(i+1)} = a_h \Rightarrow q_h(i+1) - a_h q_k(i+1) = 0 \tag{3}$$

for all $i \in \mathbb{Z}$, or more compactly as

$$\boldsymbol{e}_h^T \boldsymbol{q}(i{+}1) = 0, \tag{4}$$

where

$$\boldsymbol{e}_h^T = \begin{bmatrix} 0_1 & \cdots & 1_h & \cdots & -a_h & \cdots & 0_n \end{bmatrix}. \tag{5}$$

The task formulation above means that the problem of interest can be generally defined as a stable closed-loop system design using the linear memoryless state feedback controller of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i), \tag{6}$$

where $\boldsymbol{K} \in \mathbb{R}^{r\times n}$ is the controller feedback gain matrix, and the design constraint is considered in the general matrix equality form,

$$\boldsymbol{E}\boldsymbol{q}(i+1) = 0, \tag{7}$$

with $\boldsymbol{E} \in \mathbb{R}^{k\times n}$, $\operatorname{rank} \boldsymbol{E} = k \le r$. In general, $\boldsymbol{E}$ reflects the prescribed fixed ratio of two or more state variables.

Next, it is considered that the system is controllable and observable ($\operatorname{rank}(z\boldsymbol{I} - \boldsymbol{F}, \boldsymbol{G}) = n$, $\forall z \in \mathcal{C}$, $\operatorname{rank}(z\boldsymbol{I}-\boldsymbol{F}, \boldsymbol{C}) = n$, $\forall z \in \mathbb{C}$, respectively), and (except for Section 5.4) that all state variables are measurable.

3. Preliminaries

Proposition 1. *Let $\boldsymbol{\Gamma} \in \mathbb{R}^{n\times n}$ be a real square matrix with non-repeated eigenvalues, satisfying the equality constraint*

$$\boldsymbol{e}^T\boldsymbol{\Gamma} = \boldsymbol{0}. \tag{8}$$

Then one from its eigenvalues is zero, and (normalized) $\boldsymbol{e}^T$ is the left raw eigenvector of $\boldsymbol{\Gamma}$ associated with the zero eigenvalue.

Proof. If $\boldsymbol{\Gamma} \in \mathbb{R}^{n\times n}$ is a real square matrix satisfying the above eigenvalue properties, then the eigenvalue decomposition of $\boldsymbol{\Gamma}$ takes the form

$$\boldsymbol{\Gamma} = \boldsymbol{N}\boldsymbol{Z}\boldsymbol{M}^T, \qquad \boldsymbol{M}^T\boldsymbol{N} = \boldsymbol{I}, \tag{9}$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{n}_1 & \cdots & \boldsymbol{n}_n \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \cdots & \boldsymbol{m}_n \end{bmatrix}, \tag{10}$$

$$\boldsymbol{Z} = \mathrm{diag}\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}, \tag{11}$$

where $\boldsymbol{n}_l$ is the right eigenvector and $\boldsymbol{m}_l^T$ is the left eigenvector associated with the eigenvalue z_l of $\boldsymbol{\Gamma}$, and $\{z_l,\ l = 1, 2, \dots, n\}$ is the set of the eigenvalues of $\boldsymbol{\Gamma}$. Then (8) can be rewritten as

$$\begin{aligned}\boldsymbol{0} = \boldsymbol{d}^T &\begin{bmatrix} \boldsymbol{n}_1 \cdots \boldsymbol{n}_h \cdots \boldsymbol{n}_n \end{bmatrix}\\ &\cdot \mathrm{diag}\begin{bmatrix} z_1 \cdots z_h \cdots z_n \end{bmatrix}\boldsymbol{M}^T.\end{aligned} \tag{12}$$

If $\boldsymbol{e}^T = \boldsymbol{m}_h^T$, then orthogonal property (9) implies

$$\begin{aligned}&\boldsymbol{0}\\ &= \begin{bmatrix} 0_1 \cdots 1_h \cdots 0_n \end{bmatrix} \mathrm{diag}\begin{bmatrix} z_1 \cdots z_h \cdots z_n \end{bmatrix}\boldsymbol{M}^T,\end{aligned} \tag{13}$$

and it is evident that (13) can be satisfied only if $z_h = 0$. This concludes the proof. ■

Proposition 2. (Matrix pseudoinverse) *Let $\boldsymbol{\Theta}$ be a matrix variable and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\Lambda}$ known non-square matrices of appropriate dimensions such that the equality*

$$\boldsymbol{A}\boldsymbol{\Theta}\boldsymbol{B} = \boldsymbol{\Lambda} \tag{14}$$

is set. Then all solutions to $\boldsymbol{\Theta}$ mean that

$$\boldsymbol{\Theta} = \boldsymbol{A}^{\ominus 1}\boldsymbol{\Lambda}\boldsymbol{B}^{\ominus 1} + \boldsymbol{\Theta}^{\circ} - \boldsymbol{A}^{\ominus 1}\boldsymbol{A}\boldsymbol{\Theta}^{\circ}\boldsymbol{B}\boldsymbol{B}^{\ominus 1}, \tag{15}$$

where

$$\boldsymbol{A}^{\ominus 1} = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}, \quad \boldsymbol{B}^{\ominus 1} = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T \tag{16}$$

is a Moore–Penrose pseudoinverse of $\boldsymbol{A}$, $\boldsymbol{B}$, respectively, and $\boldsymbol{\Theta}^{\circ}$ is an arbitrary matrix of appropriate dimensions.

Proof. See, e.g., the work of Skelton *et al.* (1998, p. 13). ■

4. Enhanced representation of the BRL

The following further assumptions are imposed to obtain an enhanced LMI representation of the bounded real lemma.

Proposition 3. (Quadratic performance) *Given the stable system (1), (2), we have*

$$\sum_{l=0}^{\infty}(\boldsymbol{y}^T(l)\boldsymbol{y}(l) - \gamma\boldsymbol{u}^T(l)\boldsymbol{u}(l)) > 0, \tag{17}$$

where $\gamma > 0$ is the squared H_∞ norm of the transfer function matrix of the system.

Proof. It is evident that

$$\widetilde{\boldsymbol{y}}(z) = \boldsymbol{G}(z)\widetilde{\boldsymbol{u}}(z), \tag{18}$$

where

$$\boldsymbol{G}(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} + \boldsymbol{D} \tag{19}$$

is the discrete $m\times r$ transfer function matrix of the system (1) and (2), $\widetilde{\boldsymbol{y}}(z)$ and $\widetilde{\boldsymbol{u}}(z)$ stand for the $\mathbb{Z}$ transform of the m dimensional objective vector and the r dimensional input vector, respectively. Then (18) implies that

$$\|\widetilde{\boldsymbol{y}}(z)\| \le \|\boldsymbol{G}(z)\|\|\widetilde{\boldsymbol{u}}(z)\|, \tag{20}$$

where $\|\boldsymbol{G}(z)\|$ is the H_2 norm of the discrete transfer function matrix $\boldsymbol{G}(z)$. Since the H_∞ norm satisfies

$$\frac{1}{\sqrt{m}}\|\boldsymbol{G}(z)\|_\infty \le \|\boldsymbol{G}(z)\| \le \sqrt{r}\|\boldsymbol{G}(z)\|_\infty, \tag{21}$$

using the notation $\|\boldsymbol{G}(z)\|_\infty = \sqrt{\gamma}$, the inequality (21) can be rewritten as

$$0 < \frac{1}{\sqrt{m}} < \frac{\|\widetilde{\boldsymbol{y}}(z)\|}{\sqrt{\gamma}\|\widetilde{\boldsymbol{u}}(z)\|} \le \frac{1}{\sqrt{\gamma}}\|\boldsymbol{G}(z)\| \le \sqrt{r}. \tag{22}$$

Thus, based on Parseval's theorem, (22) yields

$$0 < \frac{\|\widetilde{\boldsymbol{y}}(z)\|}{\sqrt{\gamma}\|\widetilde{\boldsymbol{u}}(z)\|} = \frac{\Big(\sum\limits_{i=0}^{\infty}\boldsymbol{y}^T(i)\boldsymbol{y}(i)\Big)^{\frac{1}{2}}}{\sqrt{\gamma}\Big(\sum\limits_{i=0}^{\infty}\boldsymbol{u}^T(i)\boldsymbol{u}(i)\Big)^{\frac{1}{2}}}, \tag{23}$$

and subsequently

$$\sum_{i=0}^{\infty}\boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma\sum_{i=0}^{\infty}\boldsymbol{u}^T(i)\boldsymbol{u}(i) > 0. \tag{24}$$

Thus (24) implies (17). This concludes the proof. ■

Generally speaking, if it is not in contradiction with design requirements, (17) can be used to extend a Lyapunov function candidate for linear discrete-time systems.

To simplify proofs of theorems in further parts of the paper, a sketch of the proof of the BRL is presented first.

Proposition 4. (Bounded real lemma) *The autonomous system (1), (2) is stable with the quadratic performance $\|\boldsymbol{C}(z\boldsymbol{I}-\boldsymbol{F})^{-1}\boldsymbol{G}+\boldsymbol{D}\|_\infty \le \sqrt{\gamma}$ if there exist a symmetric positive definite matrix $\boldsymbol{P} > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and a positive scalar $\gamma > 0$, $\gamma \in \mathbb{R}$ such that*

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} - \boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} & \boldsymbol{C}^T\\ * & \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} - \gamma\boldsymbol{I}_r & \boldsymbol{D}^T\\ * & * & -\boldsymbol{I}_m \end{bmatrix} < 0, \tag{25}$$

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \qquad \gamma > 0, \tag{26}$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r\times r}$, $\boldsymbol{I}_m \in \mathbb{R}^{m\times m}$ are identity matrices, respectively.

In what follows, ‘$*$’ denotes the symmetric item in a symmetric matrix.

Proof. (See, e.g., Krokavec and Filasová, 2008a; Skelton *et al.*, 1998) Defining the Lyapunov function candidate as follows:

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \sum_{l=0}^{i-1}(\boldsymbol{y}^T(l)\boldsymbol{y}(l) - \gamma\boldsymbol{u}^T(l)\boldsymbol{u}(l)) > 0, \quad (27)$$

(17) implies that for such $\gamma > 0$ (27) is positive. The forward difference along a solution of the system is

$$\Delta v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i{+}1)\boldsymbol{P}\boldsymbol{q}(i{+}1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma\boldsymbol{u}^T(i)\boldsymbol{u}(i) < 0, \quad (28)$$

and using the expression of the state system (1), (2), the inequality (28) becomes

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^T(i)(\boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P} + \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F})\boldsymbol{q}(i)\\ &+ \boldsymbol{u}^T(i)(\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{F} + \boldsymbol{D}^T\boldsymbol{C})\boldsymbol{q}(i)\\ &+ \boldsymbol{q}^T(i)(\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} + \boldsymbol{C}^T\boldsymbol{D})\boldsymbol{u}(i)\\ &+ \boldsymbol{u}^T(i)(\boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} + \boldsymbol{D}^T\boldsymbol{D} - \gamma\boldsymbol{I}_r)\boldsymbol{u}(i) < 0.\end{aligned} \quad (29)$$

Thus, introducing the notation

$$\boldsymbol{q}_c^T(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{u}^T(i) \end{bmatrix}, \quad (30)$$

we obtain

$$\Delta v(\boldsymbol{q}_c(i)) = \boldsymbol{q}_c^T(i)\boldsymbol{P}_c\,\boldsymbol{q}_c(i) < 0, \quad (31)$$

where

$$\boldsymbol{P}_c = \begin{bmatrix} \boldsymbol{P}_{c11} & \boldsymbol{P}_{c12} \\ * & \boldsymbol{P}_{c22} \end{bmatrix} < 0, \quad (32)$$

$$\boldsymbol{P}_{c11} = \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} + \boldsymbol{C}^T\boldsymbol{C} - \boldsymbol{P}, \quad (33)$$

$$\boldsymbol{P}_{c12} = \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} + \boldsymbol{C}^T\boldsymbol{D}, \quad (34)$$

$$\boldsymbol{P}_{c22} = \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G} + \boldsymbol{D}^T\boldsymbol{D} - \gamma\boldsymbol{I}_r. \quad (35)$$

From (32)–(35) the following composite form can be deduced

$$\begin{bmatrix} \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F}-\boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} \\ * & \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G}-\gamma\boldsymbol{I}_r \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} < 0. \quad (36)$$

Writing

$$\begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} \\ * & \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^T \\ \boldsymbol{D}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \geq 0 \quad (37)$$

and comparing this with the matrix

$$\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}^T \\ * & \boldsymbol{0} & \boldsymbol{D}^T \\ * & * & -\boldsymbol{I}_m \end{bmatrix}, \quad (38)$$

it is evident that (37) is a Schur complement to (38). Thus, using (38), the LMI condition (36) can be written compactly as (25).

Note that, since the matrix in the upper-left-hand corner of the block matrix $\boldsymbol{\Xi}$ is a zero matrix, and $\boldsymbol{C}$, $\boldsymbol{D}$ are in general non-square matrices, $\boldsymbol{\Xi}$ is an indefinite matrix. This concludes the proof. ■

Direct application of the second Lyapunov method (Gajic and Qureshi, 1989; Mason and Shorten, 2004) and the BRL in the structure given by (26), (25) for affine uncertain systems as well as in constrained control design is in general ill conditioned owing to singular design conditions (Filasová and Krokavec, 2010; Veselý and Rosinová, 2009). To circumvent this problem, an enhanced LMI representation of the BRL is proposed, where the design condition proof is based on such a form of the BRL.

Theorem 1. (Enhanced LMI representation of the BRL) *The autonomous system (1), (2) is stable with the quadratic performance* $\|\boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} + \boldsymbol{D}\|_\infty \leq \sqrt{\gamma}$ *if there exist a symmetric positive definite matrix* $\boldsymbol{P} > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, *a square matrix* $\boldsymbol{W} \in \mathbb{R}^{n\times n}$, *and a positive scalar* $\gamma > 0$, $\gamma \in \mathbb{R}$ *such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \qquad \gamma > 0, \quad (39)$$

$$\boldsymbol{\Upsilon} = \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{F}^T\boldsymbol{W}^T & \boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{G}^T\boldsymbol{W}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{P}-\boldsymbol{W}-\boldsymbol{W}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0, \quad (40)$$

where $\boldsymbol{I}_r \in \mathbb{R}^{r\times r}$, $\boldsymbol{I}_m \in \mathbb{R}^{m\times m}$ *are identity matrices.*

Proof. Since (1) can be rewritten as

$$\boldsymbol{q}(i{+}1) - \boldsymbol{F}\boldsymbol{q}(i) - \boldsymbol{G}\boldsymbol{u}(i) = \boldsymbol{0} \quad (41)$$

with an arbitrary square matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$, we get

$$\boldsymbol{q}^T(i{+}1)\boldsymbol{X}\big(\boldsymbol{q}(i{+}1) - \boldsymbol{F}\boldsymbol{q}(i) - \boldsymbol{G}\boldsymbol{u}(i)\big) = \boldsymbol{0}. \quad (42)$$

Now, not substituting (1) into (28) but adding (42) and transposing (42) to (28) gives

$$\begin{aligned}\Delta v(\boldsymbol{q}(i)) &= \boldsymbol{q}^T(i{+}1)\boldsymbol{P}\boldsymbol{q}(i{+}1)\\ &- \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma\boldsymbol{u}^T(i)\boldsymbol{u}(i)\\ &+ \big(\boldsymbol{q}(i{+}1) - \boldsymbol{F}\boldsymbol{q}(i) - \boldsymbol{G}\boldsymbol{u}(i)\big)^T\boldsymbol{X}^T\boldsymbol{q}(i{+}1)\\ &+ \boldsymbol{q}^T(i{+}1)\boldsymbol{X}\big(\boldsymbol{q}(i{+}1) - \boldsymbol{F}\boldsymbol{q}(i) - \boldsymbol{G}\boldsymbol{u}(i)\big) < 0.\end{aligned} \quad (43)$$

Thus, with respect to (2), (43) can be rewritten as

$$\boldsymbol{q}^{\circ T}(i)\boldsymbol{J}^{\circ}\boldsymbol{q}^{\circ}(i) < 0, \tag{44}$$

where

$$\boldsymbol{q}^{\circ T}(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{u}^T(i) & \boldsymbol{q}^T(i+1) \end{bmatrix}, \tag{45}$$

$$\boldsymbol{J}^{\circ} = \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C}-\boldsymbol{P} & \boldsymbol{C}^T\boldsymbol{D} & -\boldsymbol{F}^T\boldsymbol{X}^T \\ * & \boldsymbol{D}^T\boldsymbol{D}-\gamma\boldsymbol{I}_r & -\boldsymbol{G}^T\boldsymbol{X}^T \\ * & * & \boldsymbol{P}+\boldsymbol{X}+\boldsymbol{X}^T \end{bmatrix} < 0, \tag{46}$$

or, exploiting the composite form,

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{0} & -\boldsymbol{F}^T\boldsymbol{X}^T \\ * & -\gamma\boldsymbol{I}_r & -\boldsymbol{G}^T\boldsymbol{X}^T \\ * & * & \boldsymbol{P}+\boldsymbol{X}+\boldsymbol{X}^T \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{C}^T\boldsymbol{D} & \boldsymbol{0} \\ * & \boldsymbol{D}^T\boldsymbol{D} & \boldsymbol{0} \\ * & * & \boldsymbol{0} \end{bmatrix} < 0. \tag{47}$$

Thus, equivalently, using (37), (38), and with $\boldsymbol{X} = -\boldsymbol{W}$, (47) implies (40). This concludes the proof. ■

It is evident that the Lyapunov matrix $\boldsymbol{P}$ is separated from the matrix parameters of the system $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{C}$ and $\boldsymbol{D}$, i.e., there are no terms containing the product of $\boldsymbol{P}$ and any of them. By introducing a new variable $\boldsymbol{W}$, original product forms are relaxed to new products $\boldsymbol{W}\boldsymbol{F}$ and $\boldsymbol{W}\boldsymbol{G}$, where $\boldsymbol{W}$ need not be symmetric and positive definite. Consequently, a robust BRL can be obtained to deal with linear systems with parametric uncertainties, as well as with singular system matrices.

Lemma 1. (Causal equivalence) *If there exists a positive definite symmetric matrix $\boldsymbol{P} > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, a matrix $\boldsymbol{W} \in \mathbb{R}^{n\times n}$, and a positive scalar $\gamma > 0$, $\gamma \in \mathbb{R}$ satisfying (40), then such $\boldsymbol{P} > 0$ and $\gamma > 0$ satisfy (25).*

Proof. Defining the congruence transform matrix

$$\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_r & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{G} & \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix} \tag{48}$$

and pre-multiplying the right-hand side of (40) by (48) and the left-hand side of (40) by the transpose of (48) give

$$\boldsymbol{\Upsilon}\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{F}^T\boldsymbol{W}^T\boldsymbol{F}-\boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{W}^T\boldsymbol{G} & \boldsymbol{F}^T\boldsymbol{W}^T & \boldsymbol{C}^T \\ \boldsymbol{G}^T\boldsymbol{W}\boldsymbol{F} & \boldsymbol{G}^T\boldsymbol{W}^T\boldsymbol{G}-\gamma\boldsymbol{I}_r & \boldsymbol{G}^T\boldsymbol{W}^T & \boldsymbol{D}^T \\ \boldsymbol{P}\boldsymbol{F}-\boldsymbol{W}^T\boldsymbol{F} & \boldsymbol{P}\boldsymbol{G}-\boldsymbol{W}^T\boldsymbol{G} & \boldsymbol{P}-\boldsymbol{W}-\boldsymbol{W}^T & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}, \tag{49}$$

$$\boldsymbol{\Pi} = \boldsymbol{T}_1^T\boldsymbol{\Upsilon}\boldsymbol{T}_1 = \begin{bmatrix} \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F}-\boldsymbol{P} & \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{G} & \boldsymbol{F}^T(\boldsymbol{P}-\boldsymbol{W}) & \boldsymbol{C}^T \\ \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{F} & \boldsymbol{G}^T\boldsymbol{P}\boldsymbol{G}-\gamma\boldsymbol{I}_r & \boldsymbol{G}^T(\boldsymbol{P}-\boldsymbol{W}) & \boldsymbol{D}^T \\ (\boldsymbol{P}-\boldsymbol{W}^T)\boldsymbol{F} & (\boldsymbol{P}-\boldsymbol{W}^T)\boldsymbol{G} & \boldsymbol{P}-\boldsymbol{W}-\boldsymbol{W}^T & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{I}_m \end{bmatrix}. \tag{50}$$

Using the Schur complement property, (50) can be rewritten as

$$\boldsymbol{\Pi} = \boldsymbol{\Psi} + \boldsymbol{\Phi} < 0, \tag{51}$$

where $\boldsymbol{\Psi} < 0$ is given in (25), while $\boldsymbol{\Phi} \geq 0$,

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{F}^T(\boldsymbol{P}-\boldsymbol{W}) \\ \boldsymbol{G}^T(\boldsymbol{P}-\boldsymbol{W}) \end{bmatrix} \cdot \boldsymbol{\Delta}^{-1}\begin{bmatrix} (\boldsymbol{P}-\boldsymbol{W}^T)\boldsymbol{F} & (\boldsymbol{P}-\boldsymbol{W}^T)\boldsymbol{G} \end{bmatrix} \tag{52}$$

since (40) is feasible if

$$\boldsymbol{\Delta} = -(\boldsymbol{P}-\boldsymbol{W}-\boldsymbol{W}^T) > 0. \tag{53}$$

Hence the conclusion (51)–(53) implies the proof. ■

Remark 1. It is easily verified, e.g., using (51)–(53), that (25) is equivalent to (40) if $\boldsymbol{W} = \boldsymbol{W}^T$, $\boldsymbol{P} = \boldsymbol{W} > 0$.

The state-feedback control problem is to find for an optimized (or prescribed) scalar $\gamma > 0$ the state-feedback gain $\boldsymbol{K}$ such that the control law guarantees an upper bound of $\sqrt{\gamma}$ to the H_∞ norm of the closed-loop transfer function.

Lemma 2. *The system (1), (2) under the control (6) is stable with the quadratic performance $\|\boldsymbol{C}_c(z\boldsymbol{I}-\boldsymbol{F}_c)^{-1}\boldsymbol{G}\|_\infty \leq \sqrt{\gamma}$ and $\boldsymbol{F}_c = \boldsymbol{F}-\boldsymbol{G}\boldsymbol{K}$, $\boldsymbol{C}_c = \boldsymbol{C}-\boldsymbol{D}\boldsymbol{K}$ if there exist a positive definite symmetric matrix $\boldsymbol{S} \in \mathbb{R}^{n\times n}$, a regular square matrix $\boldsymbol{V} \in \mathbb{R}^{n\times n}$, a matrix $\boldsymbol{U} \in \mathbb{R}^{r\times n}$, and a scalar $\gamma > 0$, $\gamma \in \mathbb{R}$, such that*

$$\boldsymbol{S} = \boldsymbol{S}^T > 0, \qquad \gamma > 0, \tag{54}$$

$$\begin{bmatrix} -\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{V}\boldsymbol{F}^T-\boldsymbol{U}\boldsymbol{G}^T & \boldsymbol{V}\boldsymbol{C}^T-\boldsymbol{U}\boldsymbol{D}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{G}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}-\boldsymbol{V}-\boldsymbol{V}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0. \tag{55}$$

The control law gain matrix is now given as

$$\boldsymbol{K} = \boldsymbol{U}^T\boldsymbol{V}^{-T}. \tag{56}$$

Proof. Since $\boldsymbol{W}$ is an arbitrary square matrix, $\boldsymbol{W}$ can be chosen to be regular, i.e., $\det\boldsymbol{W} \neq 0$, and the congruence transform matrix $\boldsymbol{T}_2$ can be defined as follows:

$$\boldsymbol{T}_2 = \operatorname{diag}\begin{bmatrix} \boldsymbol{W}^{-1} & \boldsymbol{I}_r & \boldsymbol{W}^{-1} & \boldsymbol{I}_m \end{bmatrix}. \tag{57}$$

Multiplying the left-hand side of (40) by $\boldsymbol{T}_2$ and the right-hand side of (40) by $\boldsymbol{T}_2^T$ gives

$$\begin{bmatrix} -\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{V}\boldsymbol{F}^T & \boldsymbol{V}\boldsymbol{C}^T \\ * & -\gamma\boldsymbol{I}_r & \boldsymbol{G}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}-\boldsymbol{V}-\boldsymbol{V}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0, \tag{58}$$

$$\boldsymbol{S} = \boldsymbol{W}^{-1}\boldsymbol{P}\boldsymbol{W}^{-T}, \quad \boldsymbol{W}^{-1} = \boldsymbol{V}. \tag{59}$$

Inserting $\boldsymbol{F} \leftarrow \boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}$, $\boldsymbol{C} \leftarrow \boldsymbol{C}_c = \boldsymbol{C} - \boldsymbol{D}\boldsymbol{K}$ into (58) gives

$$\begin{bmatrix} -\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{V}(\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K})^T & \boldsymbol{V}(\boldsymbol{C}-\boldsymbol{D}\boldsymbol{K})^T \\ * & -\gamma \boldsymbol{I}_r & \boldsymbol{G}^T & \boldsymbol{D}^T \\ * & * & \boldsymbol{S}-\boldsymbol{V}-\boldsymbol{V}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0, \tag{60}$$

and with

$$\boldsymbol{U} = \boldsymbol{V}\boldsymbol{K}^T \tag{61}$$

(60) implies (55). This concludes the proof. ■

Note that $\boldsymbol{S}$ is a symmetric Lyapunov matrix owing to the fact that $\boldsymbol{W}$ is a regular square matrix as well as that $\boldsymbol{S}$ is separated from the matrix parameters of the system.

5. Constrained control design

5.1. Constrained control. Using the control law (6), the closed-loop control equations take the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i), \tag{62}$$

$$\boldsymbol{y}(i) = (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{K})\boldsymbol{q}(i). \tag{63}$$

Given a matrix $\boldsymbol{E} \in \mathbb{R}^{k \times n}$, $\operatorname{rank}\boldsymbol{E} = k \le r$, we now consider the design constraint (7) for all nonzero natural numbers i. From Proposition 1 it is clear that such a design task is singular.

Therefore, it is supposed that $\boldsymbol{E}$ should be prescribed in such a way that the equalities

$$\boldsymbol{E}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \boldsymbol{0}, \tag{64}$$

$$\boldsymbol{E}\boldsymbol{F} = \boldsymbol{E}\boldsymbol{G}\boldsymbol{K} \tag{65}$$

can be set, as well as that the closed-loop system matrix $(\boldsymbol{F}$–$\boldsymbol{G}\boldsymbol{K})$ be stable (all its eigenvalues lie in the unit circle in the complex plane $\mathcal{Z}$).

Solving (65) with respect to $\boldsymbol{K}$, (15) implies all solutions of $\boldsymbol{K}$ as follows:

$$\boldsymbol{K} = (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{F} + (\boldsymbol{I} - (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{G})\boldsymbol{K}^{\circ}, \tag{66}$$

where $\boldsymbol{K}^{\circ}$ is an arbitrary matrix with appropriate dimension and

$$(\boldsymbol{E}\boldsymbol{G})^{\ominus 1} = (\boldsymbol{E}\boldsymbol{G})^T\big(\boldsymbol{E}\boldsymbol{G}(\boldsymbol{E}\boldsymbol{G})^T\big)^{-1}, \tag{67}$$

where $(\boldsymbol{E}\boldsymbol{G})^{\ominus 1}$ is a Moore–Penrose pseudo-inverse of $\boldsymbol{E}\boldsymbol{G}$. Thus, it is possible to express (66) as

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{L}\boldsymbol{K}^{\circ}, \tag{68}$$

where

$$\boldsymbol{J} = (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{F} \tag{69}$$

and

$$\boldsymbol{L} = \boldsymbol{I}_r - (\boldsymbol{E}\boldsymbol{G})^T\big(\boldsymbol{E}\boldsymbol{G}(\boldsymbol{E}\boldsymbol{G})^T\big)^{-1}\boldsymbol{E}\boldsymbol{G} \tag{70}$$

is the projection matrix (the orthogonal projector onto the null space $\mathcal{N}_{\boldsymbol{E}\boldsymbol{G}}$ of $\boldsymbol{E}\boldsymbol{G}$ (Krokavec and Filasová, 2008a)).

5.2. Control parameter design.

Theorem 2. *The system (1), (2) under the control (6) satisfying the constraint (7) is stable with $\|\boldsymbol{C}_c(z\boldsymbol{I} - \boldsymbol{F}_c)^{-1}\boldsymbol{G}^{\circ}\|_\infty \le \sqrt{\gamma}$, and $\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}$, $\boldsymbol{C}_c = \boldsymbol{C} - \boldsymbol{D}\boldsymbol{K}$ if there exist a positive definite symmetric matrix $\boldsymbol{S} \in \mathbb{R}^{n \times n}$, a regular square matrix $\boldsymbol{V} \in \mathbb{R}^{n \times n}$, a matrix $\boldsymbol{U} \in \mathbb{R}^{r \times n}$, and a scalar $\gamma > 0$, $\gamma \in \mathbb{R}$, such that*

$$\begin{bmatrix} -\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{V}\boldsymbol{F}^{\circ T} - \boldsymbol{U}\boldsymbol{G}^{\circ T} & \boldsymbol{V}\boldsymbol{C}^{\circ T} - \boldsymbol{U}\boldsymbol{D}^{\circ T} \\ * & -\gamma \boldsymbol{I}_r & \boldsymbol{G}^{\circ T} & \boldsymbol{D}^{\circ T} \\ * & * & \boldsymbol{S}-\boldsymbol{V}-\boldsymbol{V}^T & \boldsymbol{0} \\ * & * & * & -\boldsymbol{I}_m \end{bmatrix} < 0, \tag{71}$$

$$\boldsymbol{S} = \boldsymbol{S}^T > 0, \qquad \gamma > 0, \tag{72}$$

where

$$\boldsymbol{F}^{\circ} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{J}, \qquad \boldsymbol{G}^{\circ} = \boldsymbol{G}\boldsymbol{L}, \tag{73}$$

$$\boldsymbol{C}^{\circ} = \boldsymbol{C} - \boldsymbol{D}\boldsymbol{J}, \qquad \boldsymbol{D}^{\circ} = \boldsymbol{D}\boldsymbol{L}. \tag{74}$$

Then

$$\boldsymbol{K}^{\circ} = \boldsymbol{U}^T\boldsymbol{V}^{-T}, \quad \boldsymbol{F}_c = \boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}, \tag{75}$$

and the control law gain matrix $\boldsymbol{K}$ is given as in (68).

Proof. Substituting (6) and (68) into (1) and (2) gives

$$\boldsymbol{q}(i+1) = \boldsymbol{F}^{\circ}\boldsymbol{q}(i) + \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i), \tag{76}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}^{\circ}\boldsymbol{q}(i) + \boldsymbol{D}^{\circ}\boldsymbol{u}^{\circ}(i). \tag{77}$$

Since now (76) can be rewritten as

$$\boldsymbol{q}(i+1) - \boldsymbol{F}^{\circ}\boldsymbol{q}(i) - \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i) = \boldsymbol{0} \tag{78}$$

with an arbitrary square matrix $\boldsymbol{X} \in \mathbb{R}^{n \times n}$, this yields

$$\boldsymbol{q}^T(i+1)\boldsymbol{X}\big(\boldsymbol{q}(i+1) - \boldsymbol{F}^{\circ}\boldsymbol{q}(i) - \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i)\big) = \boldsymbol{0}. \tag{79}$$

Defining the Lyapunov function as

$$v(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \sum_{l=0}^{i-1}(\boldsymbol{y}^T(l)\boldsymbol{y}(l) - \gamma \boldsymbol{u}^{\circ T}(l)\boldsymbol{u}^{\circ}(l)) > 0, \tag{80}$$

the forward difference along a solution of the system (78), (79) is

$$\begin{aligned} &\Delta v(\boldsymbol{q}(i)) \\ &= \boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) \\ &\quad + \boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma \boldsymbol{u}^{\circ T}(i)\boldsymbol{u}^{\circ}(i) < 0. \end{aligned} \tag{81}$$

Adding (79) as well as the transpose of (79) to (81) results in

$$\begin{aligned} &\Delta v(\boldsymbol{q}(i)) \\ &= \boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1) - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) \\ &\quad + \boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma \boldsymbol{u}^{\circ T}(i)\boldsymbol{u}^{\circ}(i) \\ &\quad + \big(\boldsymbol{q}(i+1) - \boldsymbol{F}^{\circ}\boldsymbol{q}(i) - \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i)\big)^T\boldsymbol{X}^T\boldsymbol{q}(i+1) \\ &\quad + \boldsymbol{q}^T(i+1)\boldsymbol{X}\big(\boldsymbol{q}(i+1) - \boldsymbol{F}^{\circ}\boldsymbol{q}(i) - \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i)\big) < 0. \end{aligned} \tag{82}$$

Since (82) can now be compactly written as

$$\boldsymbol{q}^{\bullet T}(i)\boldsymbol{J}^{\bullet}\boldsymbol{q}^{\bullet}(i) < 0, \tag{83}$$

where

$$\boldsymbol{q}^{\bullet T}(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{u}^{\circ T}(i) & \boldsymbol{q}^T(i+1) \end{bmatrix}, \tag{84}$$

$$\boldsymbol{J}^{\bullet} = \begin{bmatrix} \boldsymbol{C}^{\circ T}\boldsymbol{C}^{\circ} - \boldsymbol{P} & \boldsymbol{C}^{\circ T}\boldsymbol{D}^{\circ} & -\boldsymbol{F}^{\circ T}\boldsymbol{X}^T \\ * & \boldsymbol{D}^{\circ T}\boldsymbol{D}^{\circ} - \gamma\boldsymbol{I}_r & -\boldsymbol{G}^{\circ T}\boldsymbol{X}^T \\ * & * & \boldsymbol{P}+\boldsymbol{X}+\boldsymbol{X}^T \end{bmatrix} < 0, \tag{85}$$

it is evident that (85) takes the same structure as (46), and so, due to (46), by replacing the matrices $(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{C}, \boldsymbol{D})$ in (55) by $(\boldsymbol{F}^{\circ}, \boldsymbol{G}^{\circ}, \boldsymbol{C}^{\circ}, \boldsymbol{D}^{\circ})$, the inequality (71) is obtained. This concludes the proof. ■

Remark 2. It is only in unforced mode that the state-variable vectors belongs to the prescribed constraint subspace $\mathcal{N}_{\boldsymbol{E}}$ given as

$$\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{E}} = \{\boldsymbol{q} : \boldsymbol{E}\boldsymbol{q} = \boldsymbol{0}\}. \tag{86}$$

Thus, the system states are constrained in this subspace (the null space of $\boldsymbol{E}$) for all nonzero natural numbers i, and stay within the constraint subspace, i.e., $\boldsymbol{F}_c\boldsymbol{q}(i) \in \mathcal{N}_{\boldsymbol{E}}$ (Ko and Bitmead, 2007a; Krokavec and Filasová, 2008b).

5.3. Constrained forced mode. The state control in a forced mode is defined by the control policy

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \boldsymbol{W}_w\boldsymbol{w}(i), \tag{87}$$

where $\boldsymbol{w}(i) \in \mathbb{R}^r$ is a desired output vector signal, and $\boldsymbol{W}_w \in \mathbb{R}^{r\times r}$ is the signal gain matrix. For the output equation of the form (2) and if the next condition is satisfied,

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = n + r, \tag{88}$$

based on the static decoupling principle, $\boldsymbol{W}_w$ can be designed as (Wang, 2003)

$$\boldsymbol{W}_w = \Big(\boldsymbol{C}\big(\boldsymbol{I}_n - (\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K})\big)^{-1}\boldsymbol{G} + \boldsymbol{D}\Big)^{-1}. \tag{89}$$

Note that the state equality constraint (64) has no direct influence on $\boldsymbol{y}(i)$.

Theorem 3. *If the closed-loop system state variables satisfy the state constraint (86), then the common state variable vector $\boldsymbol{q}_d(i) = \boldsymbol{E}\boldsymbol{q}(i)$, $\boldsymbol{q}_d(i) \in \mathbb{R}^k$ attains the steady-state value*

$$\boldsymbol{q}_d = \boldsymbol{E}\boldsymbol{W}_w\boldsymbol{w}_s. \tag{90}$$

Proof. Using the control policy (87), where $\boldsymbol{K}$ satisfies (65), we get

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}(\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}(i). \tag{91}$$

Since (91) implies

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}(i), \tag{92}$$

owing to (64), it is evident that the common state variable $q_d(i)$ of the closed-loop system in a steady state is proportional to the steady state of the desired signal $\boldsymbol{w}_s$ and takes the value (90). This concludes the proof. ■

Note that since $\boldsymbol{K}$ is optimized with respect (64), the condition (86) is not fulfilled.

5.4. Observer state feedback. The observer state feedback control law is now defined as

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}_e(i). \tag{93}$$

where $\boldsymbol{q}_e(i) \in \mathbb{R}^n$ is a state vector variable estimate. Using the standard Luenberger observer of the form

$$\boldsymbol{q}_e(i+1) = \boldsymbol{F}\boldsymbol{q}_e(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{H}(\boldsymbol{y}(i) - \boldsymbol{y}_e(i)), \tag{94}$$

$$\boldsymbol{y}_e(i) = \boldsymbol{C}\boldsymbol{q}_e(i) + \boldsymbol{D}\boldsymbol{u}(i) \tag{95}$$

the error $\boldsymbol{e}(i) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i)$ between the actual state and the estimated state at time instant i has to satisfy the autonomous difference equation

$$\boldsymbol{e}(i+1) = (\boldsymbol{A} - \boldsymbol{H}\boldsymbol{C})\boldsymbol{e}(i), \tag{96}$$

and the estimator gain matrix $\boldsymbol{H} \in \mathbb{R}^{n\times m}$ has to be designed in such a way that the observer system matrix $\boldsymbol{F}_e = \boldsymbol{F} - \boldsymbol{H}\boldsymbol{C}$ is a stable matrix.

Theorem 4. *If an observer-based control structure of the system (1), (2) is realized by the control law (93), where $\boldsymbol{K}$ is satisfied (65) and the sequence of state estimates is produced by (94), (95), then the state equality constraint (64) is fulfilled in steady state.*

Proof. Assembling the system state equation (1), (2) and the observer error dynamics (96),

$$\begin{bmatrix} \boldsymbol{q}(i+1) \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}-\boldsymbol{G}\boldsymbol{K} & \boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{H}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix} \tag{97}$$

((97) is implied by the separation principle). Defining the matrix $\boldsymbol{T}_3$ as

$$\boldsymbol{T}_3 = \begin{bmatrix} \boldsymbol{E} & \\ & \boldsymbol{I} \end{bmatrix} \tag{98}$$

and multiplying left-hand side of (97) by $\boldsymbol{T}_3$ gives

$$\begin{bmatrix} \boldsymbol{E}\boldsymbol{q}(i+1) \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}(\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K}) & \boldsymbol{E}\boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{H}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix}. \tag{99}$$

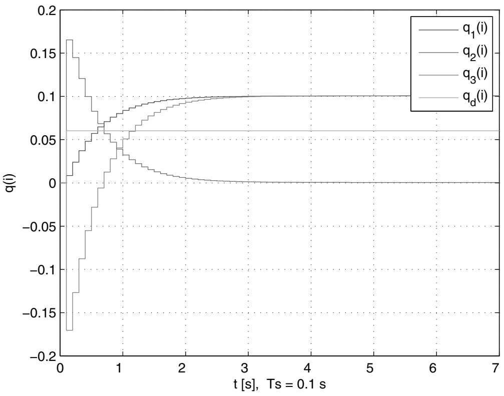

Fig. 1. Step responses of an extended set of state variables.

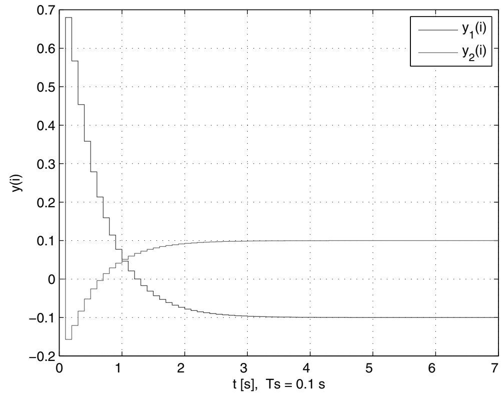

Fig. 2. System output step response.

Substituting from (64) states

$$\begin{bmatrix} \boldsymbol{E}\boldsymbol{q}(i+1) \\ \boldsymbol{e}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{E}\boldsymbol{G}\boldsymbol{K} \\ \boldsymbol{0} & \boldsymbol{F}-\boldsymbol{H}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{e}(i) \end{bmatrix}, \tag{100}$$

and it is evident that with stable $\boldsymbol{F}_e$ in steady state, i.e., when $\boldsymbol{e}(i+1) = \boldsymbol{e}(i) = \boldsymbol{0}$, such control satisfies (64). This concludes the proof. ■

Corollary 1. *It can be easily verified by straightforward calculation that, using the control law form*

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}_e(i) + \boldsymbol{W}_w\boldsymbol{w}(i), \tag{101}$$

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}\boldsymbol{G}\big(\boldsymbol{K}\boldsymbol{e}(i) + \boldsymbol{W}_w\boldsymbol{w}(i)\big), \tag{102}$$

the common state variable $q_d(i)$ of the closed-loop system in steady state is proportional to the steady state of the desired signal $\boldsymbol{w}_s$.

6. Illustrative example

To demonstrate the properties of the proposed approach, the system with two inputs and two outputs is used in the example. The parameters of the system are

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \quad \boldsymbol{D} = \boldsymbol{0},$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0010 & 0.0010 \\ 0.0206 & 0.0197 \\ 0.0077 & -0.0078 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix},$$

respectively, for the sampling period $\Delta t = 0.1$ s. The state constraint was specified as

$$\frac{q_1(t) - 0.4\, q_3(t)}{q_2(t)} = 0.1,$$

which implies

$$\boldsymbol{E} = \begin{bmatrix} 1 & -0.1 & -0.4 \end{bmatrix}$$

and subsequently yields

$$(\boldsymbol{E}\boldsymbol{G})^{\ominus 1} = \begin{bmatrix} -38.2302 \\ 19.6977 \end{bmatrix},$$

$$\boldsymbol{L} = \begin{bmatrix} 0.2098 & 0.4072 \\ 0.4072 & 0.7902 \end{bmatrix},$$

$$\boldsymbol{J} = \begin{bmatrix} -44.2102 & -11.0891 & 8.9088 \\ 22.7789 & 5.7135 & -4.5902 \end{bmatrix}.$$

Solving (71) and (72) with respect to the LMI matrix variables $\boldsymbol{S}$, $\boldsymbol{U}$, $\boldsymbol{V}$ and γ using the Self-Dual-Minimization (SeDuMi) package for Matlab (Peaucelle *et al.*, 2002), the feedback gain matrix design problem in the constrained control was solved as being feasible with the results

$$\boldsymbol{U} = \begin{bmatrix} 0.0147 & 0.0285 \\ 0.3985 & 0.7733 \\ -0.0445 & -0.0864 \end{bmatrix}, \quad \gamma = 0.7324$$

$$\boldsymbol{V} = \begin{bmatrix} 0.0043 & -0.0100 & -0.0059 \\ -0.0124 & 0.1389 & 0.1049 \\ -0.0072 & 0.0868 & 0.2694 \end{bmatrix},$$

$$\boldsymbol{S} = \begin{bmatrix} 0.0049 & -0.0121 & -0.0087 \\ -0.0121 & 0.1522 & 0.1035 \\ -0.0087 & 0.1035 & 0.3180 \end{bmatrix}.$$

Inserting $\boldsymbol{U}$ and $\boldsymbol{V}$ into (75), the feedback gain matrices were computed as follows:

$$\boldsymbol{K}^{\circ} = \begin{bmatrix} 13.4769 & 5.1827 & -1.4741 \\ 26.1567 & 10.0588 & -2.8609 \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} -30.7333 & -5.9064 & 7.4347 \\ 48.9355 & 15.7723 & -7.4511 \end{bmatrix}.$$

The control law so defined produces a stable control with the closed-loop system matrix eigenvalue spectrum

$$\rho(\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K}) = \{\, 0.0000, \quad 0.0934, \quad 0.8271 \,\}.$$

Note that one eigenvalue of $\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{GK}$ is zero ($\text{rank}(\boldsymbol{E}) = 1$)) as Proposition 1 implies that such a constrained design task is a singular problem .

Simulation results for the closed-loop system response in the forced mode are presented in Fig. 1 for the state variable response and Fig. 2 for the output variable response. The initial state vector, the desired steady-state values of the output variables, and the signal gain matrix were set as $\boldsymbol{q}(0) = \boldsymbol{0}$,

$$\boldsymbol{w}(i) = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix},$$

$$\boldsymbol{W} = \begin{bmatrix} -6.2124 & -19.5270 \\ 6.6807 & 37.3138 \end{bmatrix}.$$

It is clear that the condition (92) is satisfied at all time instants except zero in such a way that

$$\boldsymbol{q}_d = E\boldsymbol{W}_w \boldsymbol{w}_s = 0.0601$$

(cf. the step responses of an extended set of state variables, including also $q_d(i)$, in Fig. 1).

7. Concluding remarks

In this paper we developed a new method based on a classical memoryless feedback H_∞ control of discrete-time systems if equality constraints tying together state variables were prescribed. The quadratic stability of the control scheme is established in the sense of an enhanced representation of the BRL to circumvent an ill-conditioned singular design task. Such a matrix inequality is linear with respect to the system variables and does not involve any product of the Lyapunov matrix and the system matrices. This provides one way for determination of a parameter-independent Lyapunov function by solving singular LMI problems. The proposed method formulates the problem as a stabilization one with a static output feedback controller whose gain takes no special structure. Compared with the author's previous results, the number of assumptions is reduced while a singular solution is guaranteed and no iteration steps are needed. This formulation allows us to find a solution to the control law without restrictive assumptions and additional specifications on the design parameters implied, e.g., by Finsler's lemma. It is clear, however, that from Theorem 3 the control law just found solves the problem even in the forced mode. The validity of the proposed method is demonstrated by a numerical example with the equality constraint tying together all state variables in a prescribed way.

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