Int. J. Appl. Math. Comput. Sci., 2014, Vol. 24, No. 4, 723–733
DOI: 10.2478/amcs-2014-0053

ON AN INFINITE DIMENSIONAL LINEAR–QUADRATIC PROBLEM WITH FIXED ENDPOINTS: THE CONTINUITY QUESTION

K. MACIEJ PRZYŁUSKI

Department of Applied Sciences
Collegium Mazovia, ul. Sokołowska 116, 08-110 Siedlce, Poland
e-mail: k.m.przyluski@gmail.com

In a Hilbert space setting, necessary and sufficient conditions for the minimum norm solution u to the equation $Su = Rz$ to be continuously dependent on z are given. These conditions are used to study the continuity of minimum energy and linear-quadratic control problems for infinite dimensional linear systems with fixed endpoints.

Keywords: minimum norm problem, linear-quadratic control, linear-quadratic economies, controllability, continuity of optimal control.

1. Introduction

The existing theory of linear-quadratic problems has been successfully applied to the design of many industrial and military control systems (see, e.g., Athans, 1971). A stochastic version of this problem plays today an important role in macroeconomics, where the so-called linear-quadratic economies are considered (see, e.g., Ljungqvist and Sargent, 2004; Sent, 1998). These (dynamic stochastic) optimizing models had to have *linear* constraints with *quadratic* objective functions to get a linear decision rule (see, e.g., Chow, 1976; Kendrick, 1981). However, such stochastic problems are frequently infinite dimensional (see, e.g., the work of Federico (2011) and the references cited therein).

We will consider infinite dimensional linear control systems which can be represented by two linear continuous operators describing the influence of control, and the constraints imposed on all of the system's trajectories by given initial and final conditions. The minimum energy and linear-quadratic problems for such systems will be developed. These problems can be studied in an appropriate Hilbert space setting. Then (as is well known) the existence and uniqueness of optimal solutions to the above problems can be easily established, under rather mild assumptions.

The purpose of our paper is to explore the conditions under which the solutions to the above-mentioned optimization problems continuously depend on initial and final conditions. Not surprisingly, these continuity (or discontinuity) conditions are strongly related to some concepts of controllability for infinite dimensional (linear) systems. The importance of the continuous dependence of the optimal solution upon the imposed initial and final conditions is obvious, in particular when developing numerical methods for the minimum energy or linear quadratic problem. For infinite dimensional linear control systems, the continuous dependence of optimal solutions on constraints on values of admissible controls has been considered by Przyłuski (1981). A much more general approach to such problems is presented by Kandilakis and Papageorgiou (1992) as well as Papageorgiou (1991).

The paper is organized as follows. In Sections 2 and 3 we consider quite general minimum norm problems. The obtained results are next applied (Section 4) to study a linear-quadratic problem. In the last sections (5 and 6) the minimum energy problem with fixed endpoints for some classes of linear infinite dimensional (discrete-time and continuous-time) control systems is considered.

The notation used in the paper is standard (see, e.g., Aubin, 2000; Laurent, 1972; Luenberger, 1969; Corless and Frazho, 2003). In particular, for any unitary space $\mathcal{H}$, and $x, y \in \mathcal{H}$, we usually denote by $(x\,|\,y)$ the inner product of x and y. Let us recall that the norm $\|x\|$ of any $x \in \mathcal{H}$ is defined as the square root of $(x\,|\,x)$. When M is a subset of a unitary space, $\overline{M}$ denotes the closure of M. For any linear subspace $\mathcal{S}$ of $\mathcal{H}$, we denote by $\mathcal{S}^\perp$ the orthogonal complement of $\mathcal{S}$. For arbitrary unitary spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we write $\mathcal{H}_1 \oplus \mathcal{H}_2$ for the

Hilbert sum of these spaces. For $h := (h_1, h_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, the norm $\|h\| := \left(\|h_1\|^2 + \|h_2\|^2\right)^{1/2}$. We shall write $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ for the (naturally) normed space of all continuous linear operators $\mathcal{H}_1 \to \mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2$, the symbol $\mathcal{L}(\mathcal{H}_1)$ is used instead of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. For any operator $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\|A\|$ denotes its (operator) norm, $\mathsf{Ker}\, A$ denotes its kernel, and $\mathsf{Im}\, A$ is its image. The (Hilbert space) adjoint of A is denoted by $A^\star$. For any Hilbert space $\mathcal{H}$ we write $\ell^2_\tau(\mathcal{H})$ for the Hilbert space of all $\mathcal{H}$-valued sequences $h = (h_k)_{k=0}^{\tau-1}$, the space being normed by the norm $|\cdot|_2$ defined (as usual) by the formula $|h|_2 := \left(\sum_{k=0}^{\tau-1}\|h_k\|^2\right)^{1/2}$.

2. Minimum norm problem

Let $\mathcal{H}_u$, $\mathcal{H}_v$ and $\mathcal{H}_z$ be real Hilbert spaces. Let $S \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_v)$ and $R \in \mathcal{L}(\mathcal{H}_z, \mathcal{H}_v)$ be fixed operators. We consider the following **minimum norm problem**.

For a given $z \in \mathcal{H}_z$, find $\widehat{u} \in \mathcal{H}_u$ such that

$$S\widehat{u} = Rz \tag{1a}$$

and

$$\|\widehat{u}\| = \inf_u \{\|u\| \mid Su = Rz\}. \tag{1b}$$

We summarize below some *well known results* concerning the above described optimization problem. We first define the **space $\mathcal{Z}$ of admissible values of z** in the following way:

$$\mathcal{Z} := \{z \in \mathcal{H}_z \mid \exists\, u \in \mathcal{H}_u\colon Su = Rz\}. \tag{2}$$

Of course, $\mathcal{Z} = R^{-1}(\mathsf{Im}\, S)$ (the inverse image of $\mathsf{Im}\, S$ under R).

Let P denote the orthogonal projection of $\mathcal{H}_u$ onto $(\mathsf{Ker}\, S)^\perp$. Assume $z \in \mathcal{Z}$ is *fixed*, and let u' and u'' be such that $Su' = Su'' = Rz$. Then $SPu' = SPu'' = Rz$. In particular, $Pu' - Pu'' \in \mathsf{Ker}\, S$, and therefore $Pu' = Pu''$. It follows that Pu is the same for all $u \in \mathcal{H}_u$ satisfying the constraint $Su = Rz$, with fixed $z \in \mathcal{Z}$. For any $z \in \mathcal{Z}$, we denote such Pu by $\widehat{u}(z)$. Observe that, for any u satisfying $Su = Rz$, we have $u = \widehat{u}(z) + (I - P)u$, where I denotes the identity operator on $\mathcal{H}_u$. It follows that

$$\|u\|^2 = \|\widehat{u}(z)\|^2 + \|(I-P)u\|^2 \ge \|\widehat{u}(z)\|^2.$$

Hence, *for any $z \in \mathcal{Z}$, $\widehat{u}(z)$ is the (unique) solution to our minimum norm problem.*

The considerations presented above show that one can define a mapping $\mathcal{Z} \to \mathcal{H}_u$, which maps $z \in \mathcal{Z}$ to the minimum norm solution $\widehat{u}(z)$ to the equation $Su = Rz$. We denote this mapping by K. The following result is well known (see, e.g., Aubin, 2000; Laurent, 1972).

Proposition 1. *The mapping $K : \mathcal{Z} \to \mathcal{H}_u$ is linear, i.e., $K(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 K z_1 + \alpha_2 K z_2$.*

Proof. Let $z_1, z_2 \in \mathcal{Z}$, $\alpha_1, \alpha_2 \in \mathbb{R}$, and $z = \alpha_1 z_1 + \alpha_2 z_2$. Since $\mathcal{Z}$ is a linear subspace of $\mathcal{H}_z$, $z \in \mathcal{Z}$. To justify that K is linear, we should prove that $\widehat{u}(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 \widehat{u}(z_1) + \alpha_2 \widehat{u}(z_2)$. To this end, let us observe that

$$\begin{aligned} &S(\alpha_1 \widehat{u}(z_1) + \alpha_2 \widehat{u}(z_2)) \\ &\quad = \alpha_1 S\widehat{u}(z_1) + \alpha_2 S\widehat{u}(z_2) \\ &\quad = \alpha_1 R z_1 + \alpha_2 R z_2 \\ &\quad = R(\alpha_1 z_1 + \alpha_2 z_2) = Rz. \end{aligned}$$

Since

$$\alpha_1 \widehat{u}(z_1) + \alpha_2 \widehat{u}(z_2) \in (\mathsf{Ker}\, S)^\perp,$$

we conclude that

$$\widehat{u}(z) = \alpha_1 \widehat{u}(z_1) + \alpha_2 \widehat{u}(z_2).$$

■

The main result of this section is the following theorem.

Theorem 1. *K is continuous if and only if the space $\mathcal{Z}$ of admissible values of z is closed in $\mathcal{H}_z$.*

Proof.
(Necessity) Let $z \in \overline{\mathcal{Z}}$, the closure of $\mathcal{Z}$. Then there exists a sequence $(z_n)_{n=1}^\infty$ such that $z_n \in \mathcal{Z}$ and $\lim z_n = z$. Let $u_n = K z_n$. Of course, $S u_n = R z_n$. Then

$$\|u_n - u_m\| \le \|K\| \|z_n - z_m\|,$$

and (since $(z_n)_{n=1}^\infty$ is convergent), $(u_n)_{n=1}^\infty$ is a Cauchy sequence, and therefore the sequence $(u_n)_{n=1}^\infty$ is also convergent. Let $u = \lim u_n$. If we take the limits of both the sides of the equality $Su_n = Rz_n$ as $n \to \infty$, we find that $Su = Rz$. This means that $z \in \mathcal{Z}$.

(Sufficiency) Let $\mathcal{Z}$ be closed. Then $\mathcal{Z}$ is a Hilbert space with respect to the inner product induced from $\mathcal{H}_z$. Let $\widetilde{R}$ denote the restriction of the operator R to the Hilbert space $\mathcal{Z}$. Observe that $\mathsf{Im}\, S \supset \mathsf{Im}\, \widetilde{R}$. Using the Douglas factorization theorem (see, e.g., Douglas, 1966; Rolewicz, 1987), we conclude that there exists an operator $\widetilde{K} \in \mathcal{L}(\mathcal{Z}, \mathcal{H}_u)$ such that $S\widetilde{K} = \widetilde{R}$. Let P denote (as usual) the orthogonal projection of $\mathcal{H}_u$ onto $(\mathsf{Ker}\, S)^\perp$. Then, for $z \in \mathcal{Z}$, $S(P\widetilde{K})z = S\widetilde{K}z = \widetilde{R}z = Rz$. Since $P\widetilde{K}z \in (\mathsf{Ker}\, S)^\perp$, $K := P\widetilde{K}$ is the mapping which assigns any $z \in \mathcal{Z}$ the minimum norm solution $\widehat{u}(z)$ to the equation $Su = Rz$. It is obvious that $K \in \mathcal{L}(\mathcal{Z}, \mathcal{H}_u)$. In particular, K is continuous. ■

Remark 1. The existing proofs of the Douglas factorization theorem are usually based on the closed graph theorem (see, e.g., Douglas, 1966; Rolewicz, 1987). So it is not surprising that to prove the sufficiency part of

Theorem 1 we could have used (instead of the Douglas factorization theorem) the closed graph theorem.

Using Theorem 1 one can prove[1] the following remarkable characterization of the closedness of the space $\mathcal{Z}$ of admissible values of z.

Corollary 1. *The following statements are equivalent:*

(i) *The space $\mathcal{Z}$ of admissible values of z is closed in $\mathcal{H}_z$.*

(ii) *There exists $\alpha \geq 0$ such that, for every $z \in \mathcal{Z}$, one can find $u \in \mathcal{H}_u$ satisfying $Su = Rz$ and the inequality $\|u\| \leq \alpha\|z\|$.*

(iii) *For every $\varepsilon > 0$, $z \in \mathcal{Z}$, and $u \in \mathcal{H}_u$ satisfying $Su = Rz$, there exists $\delta > 0$ such that for every z' satisfying the inequality $\|z - z'\| \leq \delta$ and belonging to $\mathcal{Z}$, one can find $u' \in \mathcal{H}_u$ such that, $Su' = Rz'$ and $\|u - u'\| \leq \varepsilon$.*

We see that it is important to know when the space $\mathcal{Z}$ is closed. We collect below a few simple results in this direction.

Proposition 2. $\operatorname{Im} S \supset \operatorname{Im} R$ *if and only if* $\mathcal{Z} = \mathcal{H}_z$.

In particular, if $\operatorname{Im} S \supset \operatorname{Im} R$, the space $\mathcal{Z}$ of admissible values of z is closed in $\mathcal{H}_z$.

Before formulating our next result, we recall that a linear continuous operator acting between Hilbert spaces possesses a linear continuous right inverse if and only if this operator is surjective (employ the Douglas factorization theorem or see, e.g., the work of Aubin (2000)). Let us also recall that, for any mapping L and any subset M of its domain, $L^{-1}(M)$ denotes the inverse image of M under the mapping L.

Proposition 3. *Let R be right invertible. Assume that $\mathcal{Z}$ is closed. Then $\operatorname{Im} S$ is also closed.*

Proof. Let J be a right inverse of R, so that $RJ = I$, the identity operator on $\mathcal{H}_v$. Then $\operatorname{Im} S = (RJ)^{-1}(\operatorname{Im} S) = J^{-1}\left[R^{-1}(\operatorname{Im} S)\right] = J^{-1}(\mathcal{Z})$. Since J is continuous, $J^{-1}(\mathcal{Z})$ (being equal to $\operatorname{Im} S$) is closed. ■

Remark 2. The above proposition says that *when R is right invertible and $\operatorname{Im} S \neq \overline{\operatorname{Im} S}$, the space $\mathcal{Z}$ of admissible values of z cannot be closed, and therefore the corresponding linear mapping K is discontinuous.*

Proposition 4. *Assume that $\operatorname{Im} S$ is closed. Then $\mathcal{Z}$ is closed.*

Let us note that the space $\mathcal{Z}$ of admissible values of z is always closed, when $\operatorname{Im} S$ is finite dimensional (or finite codimensional).

We end this section with the following two general remarks.

[1] Since we will not need this result, its proof is omitted.

Remark 3. Let us recall (see, e.g., Luenberger, 1969) that the *Moore–Penrose pseudoinverse* $S^\dagger$ of S exists if and only if the image of S is closed. The assumption that $\operatorname{Im} S = \overline{\operatorname{Im} S}$ significantly simplifies the minimum norm problem since then the mapping K which maps $z \in \mathcal{Z}$ to the minimum norm solution $\widehat{u}(z)$ to the equation $Su = Rz$ is equal to the restriction of the continuous linear operator $S^\dagger R$ to the (closed) subspace $\mathcal{Z}$ of $\mathcal{H}_z$.

Remark 4. Consider the special case where $\mathcal{H}_z = \mathcal{H}_v$ and $R = I$, the identity operator. Assume that $\operatorname{Im} S$ is a proper dense subspace of $\mathcal{H}_z$ (i.e., $\overline{\operatorname{Im} S} = \mathcal{H}_v \neq \operatorname{Im} S$). Then, only for $v \in \operatorname{Im} S$, there exists a (unique) solution to our minimum norm problem. When $v \notin \operatorname{Im} S$, one can consider a *relaxation* of this problem. One of the possible approaches is to solve the (unconstrained) problem of minimizing $\|u\|^2 + \rho\|Su - v\|^2$, for large positive ρ. Another possibility is to study the (constrained) minimization problem of finding $u \in \mathcal{H}_u$ of minimal norm and such that $\|Su - v\| \leq \eta$, for small positive η. These approaches are closely related. For details, the interested reader should consult Kobayashi (1978) or Emirsajłow (1989).

3. Extended minimum norm problem

Let $\mathcal{H}_0$ be a real Hilbert space and $R_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_v)$ be a given operator. We consider below the following **extended minimum norm problem**.

For given $z_0 \in \mathcal{H}_0$ and $z_v \in \mathcal{H}_v$, find $\widehat{u} \in \mathcal{H}_u$ such that

$$S\widehat{u} = R_0 z_0 + z_v \tag{3a}$$

and

$$\|\widehat{u}\| = \inf_u \{\|u\| \mid Su = R_0 z_0 + z_v\}. \tag{3b}$$

One can reduce the above problem to the minimum norm one defined by the relations (1). To this end, let I denote the identity operator on $\mathcal{H}_v$ and $\mathcal{H}_z := \mathcal{H}_0 \oplus \mathcal{H}_v$ (as usual, $\oplus$ denotes the direct sum of Hilbert spaces). Let $z = (z_0, z_v)$ and $R = \begin{bmatrix} R_0 & I \end{bmatrix}$, so that $Rz = R_0 z_0 + z_v$, and $R \in \mathcal{L}(\mathcal{H}_z, \mathcal{H}_v)$. We see at once that the relations (3) can be rewritten in the form used to define our standard minimum norm problem, with R as above. Note that, for the extended minimum norm problem, by the space of admissible values of z we should mean the following subspace of $\mathcal{H}_0 \oplus \mathcal{H}_v$:

$$\mathcal{Z} = \Big\{(z_0, z_v) \in \mathcal{H}_0 \oplus \mathcal{H}_v \mid \\ \exists\, u \in \mathcal{H}_u\colon Su = Rz_0 + z_v\Big\}.$$

Proposition 5. *The space $\mathcal{Z}$ described above is closed if and only if* $\operatorname{Im} S$ *is closed.*

Proof. We know from Proposition 4 that $\mathcal{Z}$ is closed, if $\operatorname{Im} S$ is closed. Assume now that $\operatorname{Im} S$ is closed. Since $R = \begin{bmatrix} R_0 & I \end{bmatrix}$ is right invertible, one can use Proposition 3 to deduce that $\mathcal{Z}$ is closed. ■

Proposition 6. *Let $R = \begin{bmatrix} R_0 & I \end{bmatrix}$. Assume* $\operatorname{Im} S \supset \operatorname{Im} R_0$. *Then $(z_0, z_v) \in \mathcal{Z}$ if and only if $z_v \in \operatorname{Im} S$.*

Proof. Let $z_v \in \operatorname{Im} S$. Then $z_v = S u_v$, for some $u_v \in \mathcal{H}_u$. Let $z_0 \in \mathcal{H}_0$. Since $\operatorname{Im} S \supset \operatorname{Im} R_0$, one can find $u_0 \in \mathcal{H}_u$ such that $R_0 z_0 = S u_0$. Hence $S(u_0 + u_v) = R_0 z_0 + z_v$. It follows that any $z = (z_0, z_v)$ with $z_v \in \operatorname{Im} S$ belongs to $\mathcal{Z}$.

Conversely, let $(z_0, z_v) \in \mathcal{Z}$ so that $Su = R_0 z_0 + z_v$, for some $u \in \mathcal{H}_u$. Since $\operatorname{Im} S \supset \operatorname{Im} R_0$, one can find $u_0 \in \mathcal{H}_u$ such that $R_0 z_0 = S u_0$. Then $S(u - u_0) = z_v$, i.e., $z_v \in \operatorname{Im} S$. ■

Corollary 2. *Let $R = \begin{bmatrix} R_0 & I \end{bmatrix}$. Then* $\operatorname{Im} S \supset \operatorname{Im} R_0$ *if and only if $\mathcal{Z} = \mathcal{H}_0 \oplus \operatorname{Im} S$. In particular, $\mathcal{Z} = \mathcal{H}_0 \oplus \mathcal{H}_v$ if and only if S is surjective.*

We know that, *for any $z \in \mathcal{Z}$, there exists a (uniquely defined) solution $\widehat{u}(z)$ to the extended minimum norm problem considered*. Since $z = (z_0, z_v)$, we also write $\widehat{u}(z_0, z_v)$ instead of $\widehat{u}(z)$. By virtue of Proposition 1, the mapping $(z_0, z_v) \mapsto \widehat{u}(z_0, z_v)$ is linear. It is a consequence of Theorem 1 and Proposition 5 that *this mapping is continuous if and only if* $\operatorname{Im} S$ *is closed*.

Unfortunately, the assumption that $\operatorname{Im} S$ is closed is rather restrictive. Our next result deals with the extended minimum norm problem for S whose image is *not* closed.

Theorem 2. *Assume that*

$$\operatorname{Im} S \supset \operatorname{Im} R_0 \quad \text{and} \quad \operatorname{Im} S \neq \overline{\operatorname{Im} S}.$$

Let $\widehat{u}(z_0, z_v)$ be the solution to the extended norm minimization problem. Then

$$\widehat{u}(z_0, z_v) = K_0 z_0 + K_v z_v,$$

where K_0 is linear and continuous (i.e., $K_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_u)$), and $K_v : \operatorname{Im} S \to \mathcal{H}_u$ is linear, but it cannot be continuous.

Proof. In view of Corollary 2, $\widehat{u}(z_0, z_v)$ is well-defined for all pairs (z_0, z_v) such that $z_0 \in \mathcal{H}_0$ and $z_v \in \operatorname{Im} S$. In particular, $(z_0, 0)$ and $(0, z_v)$ are in $\mathcal{Z}$. Observe that $\widehat{u}(z_0, 0)$ is the minimum norm solution to the equation $Su = R_0 z_0$, whereas $\widehat{u}(0, z_v)$ is the minimum norm solution to the equation $Su = z_v$. Since $\widehat{u}(z_0, 0)$ and $\widehat{u}(0, z_v)$ belong to $(\operatorname{Ker} S)^{\perp}$, and

$$S\left(\widehat{u}(z_0, 0) + \widehat{u}(0, z_v)\right) = R_0 z_0 + z_v,$$

we have the equality $\widehat{u}(z_0, 0) + \widehat{u}(0, z_v) = \widehat{u}(z_0, z_v)$. This means that

$$K_0 z_0 = \widehat{u}(z_0, 0), \quad K_v z_v = \widehat{u}(0, z_v).$$

The inclusion $\operatorname{Im} S \supset \operatorname{Im} R_0$ implies (see Proposition 2) that $R_0^{-1}(\operatorname{Im} S) = \mathcal{H}_0$, and therefore K_0 is continuous. On the other hand, since $\operatorname{Im} S \neq \overline{\operatorname{Im} S}$, K_v is discontinuous, in view of Remark 2. ■

4. Linear-quadratic problem

Let $\mathcal{H}_w$, $\mathcal{H}_y$ be a real Hilbert space, and $W \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_w)$, $L_1 \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_y)$, $L_2 \in \mathcal{L}(\mathcal{H}_z, \mathcal{H}_y)$ be given operators. We always assume that W *is an injection with closed image*. For Hilbert spaces, such operators are characterized (see, e.g., Aubin, 2000) by the existence of a positive constant γ such that $\|Wu\| \geq \gamma \|u\|$, for all u. This inequality is equivalent to the positive definiteness (also called coerciveness) of the self-adjoint operator $W^\star W$. It follows that W is an injection with closed image if and only if $W^\star W$ is positive definite. Since $W^\star W$ is always nonnegative definite, $W^\star W$ is positive definite if and only if the operator is invertible.

In this section we consider the following **linear quadratic problem**.

For a given $z \in \mathcal{H}_z$, find $\widehat{u} \in \mathcal{H}_u$ such that

$$S\widehat{u} = Rz \tag{4a}$$

and

$$\begin{aligned} &\|W\widehat{u}\|^2 + \|L_1 \widehat{u} + L_2 z\|^2 \\ &\quad = \inf_u \left\{ \|Wu\|^2 + \|L_1 u + L_2 z\|^2 \mid Su = Rz \right\}. \end{aligned} \tag{4b}$$

Let us observe that, for any $u \in \mathcal{H}_u$ and $z \in \mathcal{H}_z$,

$$\begin{aligned} &\|Wu\|^2 + \|L_1 u + L_2 z\|^2 \\ &\quad = \big(u \,|\, (W^\star W + L_1^\star L_1) u\big) + 2\big(u \,|\, L_1^\star L_2 z\big) \\ &\qquad + \|L_2 z\|^2. \end{aligned} \tag{5}$$

Let

$$Q := W^\star W + L_1^\star L_1.$$

Of course, $Q \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_u)$. Since W is an injection with closed image, the operator Q above defined is always (i.e., independently of L_1) positive definite, hence invertible. Moreover, there exists a unique positive definite square root $Q^{1/2}$ of Q. Observe that the first term on the right-hand side of (5) can be written as $\|Q^{1/2} u\|^2$. Since $Q^{1/2}$ is positive definite, it is also invertible. The inverse of $Q^{1/2}$ will be denoted by $Q^{-1/2}$.

Our purpose is to reduce the linear quadratic problem considered into a norm minimization one. To this end, let us compute the norm of $Q^{1/2}(u + Q^{-1} L_1^\star L_2 z)$. After easy calculations we obtain the following equality:

$$\begin{aligned}&\|Q^{1/2}(u+Q^{-1}L_1^\star L_2 z)\|^2\\&=\|Q^{1/2}u\|^2+2\big(u\,|\,L_1^\star L_2 z\big)\\&\quad+\|Q^{-1/2}L_1^\star L_2 z\|^2.\end{aligned}\tag{6}$$

It follows (cf. Eqns. (5) and (6)) that

$$\begin{aligned}\big(\|Wu\|^2+\|L_1u+L_2z\|^2\big)-\|Q^{1/2}(u+Q^{-1}L_1^\star L_2 z)\|^2\\=\|L_2z\|^2-\|Q^{-1/2}L_1^\star L_2 z\|^2.\end{aligned}$$

We see that the *difference between* $\|Wu\|^2+\|L_1u+L_2z\|^2$ *and* $\|Q^{1/2}(u+Q^{-1}L_1^\star L_2z)\|^2$ *is independent of* u. This means that, instead of the linear-quadratic problem defined by (4), one can consider the problem in which (for fixed z) we are minimizing with respect to u (for $u\in\mathcal{H}_u$ satisfying $Su=Rz$) the norm

$$\|Q^{1/2}(u+Q^{-1}L_1^\star L_2 z)\|.\tag{7}$$

Let

$$q:=u+Q^{-1}L_1^\star L_2 z.\tag{8}$$

Then (7) takes the form $\|Q^{1/2}q\|$, and the constraint $Su=Rz$ should be replaced by the equality $Sq=(R-Q^{-1}L_1^\star L_2)z$. Now, let us define on $\mathcal{H}_u$ a new inner product $(\cdot\,|\,\cdot)_Q$ by the formula $(x\,|\,y)_Q:=(x\,|\,Qy)$, where $x,y\in\mathcal{H}_u$, and $(\cdot\,|\,\cdot)$ is the original inner product of $\mathcal{H}_u$. Since Q is a positive definite operator, $(x\,|\,y)_Q$ is a well-defined inner product on $\mathcal{H}_u$. For the norm $\|\cdot\|_Q$ induced by this inner product, we have $\|q\|_Q=\|Q^{1/2}q\|$, for all $q\in\mathcal{H}_u$. Since Q is positive definite, the *norms* $\|\cdot\|_Q$ *and* $\|\cdot\|$ *(i.e., the original norm of* $\mathcal{H}_u$*) are equivalent*. Let us recall that the continuity of functions defined on $\mathcal{H}_u$ and the closedness of subsets of $\mathcal{H}_u$ are independent of the assumed norms on $\mathcal{H}_u$, if these norms are equivalent.

On account of the discussion presented above, one can formulate a minimum norm problem reflecting all the properties of the linear quadratic problem studied in this section as follows.

For a given $z\in\mathcal{H}_z$*, find* $\widehat{q}\in\mathcal{H}_u$ *such that*

$$S\widehat{q}=(R-SQ^{-1}L_1^\star L_2)z\tag{9a}$$

and

$$\|\widehat{q}\|_Q=\inf_q\left\{\|q\|_Q\mid Sq=(R-SQ^{-1}L_1^\star L_2)z\right\},\tag{9b}$$

where $Q=W^\star W+L_1^\star L_1$*, and* W *is an injection with closed image.*

It is immediate that, for a given z, the above minimum norm problem has a solution if and only if our original linear-quadratic problem defined by the relations (4) is solvable. Then the solutions $\widehat{q}$ and $\widehat{u}$ to these problems are related by (8).

Let, for the minimum norm problem defined by (9), $\mathcal{Z}_q$ denote the counterpart of the space $\mathcal{Z}$ of admissible values of z, defined in Section 1 by (2), i.e.,

$$\mathcal{Z}_q:=\left\{z\in\mathcal{H}_z\mid\exists\,q\in\mathcal{H}_u\colon Sq=(R-SQ^{-1}L_1^\star L_2)z\right\}.$$

From our deliberations in Section 1 it follows that, for every $z\in\mathcal{Z}_q$, there exists a uniquely defined solution $\widehat{q}$ to the minimum norm problem (9), and $\widehat{q}$ is a linear function of z. This function, to be denoted by K_q, is a continuous function $\mathcal{Z}_q\to\mathcal{H}_u$ if and only if $\mathcal{Z}_q$ is closed in $\mathcal{H}_z$ (see Theorem 1).

It happens that $\mathcal{Z}_q$ *is closed in* $\mathcal{H}_z$ *if and only if* $\mathcal{Z}=R^{-1}(\operatorname{Im}S)$ *is closed*. More precisely, we can prove the following elementary result, saying in particular that $\mathcal{Z}_q=\mathcal{Z}$.

Proposition 7. *For any linear mapping* $F:\mathcal{H}_z\to\mathcal{H}_u$*,*

$$R^{-1}(\operatorname{Im}S)=\big(R+SF\big)^{-1}(\operatorname{Im}S).$$

Proof. Of course, $z\in R^{-1}(\operatorname{Im}S)$ if and only if there exist u such that $Su=Rz$. Then $Su+SFz=Rz+SFz$, and $S(u+Fz)=(R+SF)z$. Now it is obvious that $z\in\big(R+SF\big)^{-1}(\operatorname{Im}S)$.

Conversely, assume that $z\in\big(R+SF\big)^{-1}(\operatorname{Im}S)$. Then there exists u such that $Su=(R+SF)z$. Then $S(u-Fz)=Rz$, and therefore $z\in R^{-1}(\operatorname{Im}S)$. ■

It should be clear now that the linear-quadratic problem studied in this section possesses a solution if and only if $z\in\mathcal{Z}=R^{-1}(\operatorname{Im}S)$. The solution is uniquely determined by z, and will be denoted (as usual) by $\widehat{u}(z)$. Let $K:\mathcal{Z}:\to\mathcal{H}_u$ be the mapping $z\mapsto\widehat{u}(z)$. From (8) we conclude that

$$K=K_q-Q^{-1}L_1^\star L_2,$$

and the linearity of K is obvious. Moreover, we are thus led to the following strengthening of Theorem 1.

Theorem 3. *Consider the linear quadratic problem defined by the relations (4). Assume that* W *is an injection with closed image. Then the linear mapping* $K:\mathcal{Z}\to\mathcal{H}_u$ *given above is (well defined and) continuous if and only if* $\mathcal{Z}=R^{-1}(\operatorname{Im}S)$ *is closed in* $\mathcal{H}_z$*.*

One can also generalize Theorem 2.

Theorem 4. *Consider the linear quadratic problem defined by the relations (4), with* $R=\begin{bmatrix}R_0 & I\end{bmatrix}$ *(see Section 3). Let* W *be an injection with closed image. Assume also that*

$$\operatorname{Im}S\supset\operatorname{Im}R_0\quad\textit{and}\quad\operatorname{Im}S\neq\overline{\operatorname{Im}S}.$$

Let $\widehat{u}(z_0, z_v)$ be the solution to the linear quadratic problem considered. Then (as in Theorem 2)

$$\widehat{u}(z_0, z_v) = K_0 z_0 + K_v z_v,$$

where K_0 is linear and continuous (i.e., $K_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_u)$), and $K_v : \operatorname{Im} S \to \mathcal{H}_u$ is linear, but it cannot be continuous.

Remark 5. The fact that any linear-quadratic problem can be reduced to an appropriate minimum norm one is well known for *control systems described by differential equations*. This reduction requires solving a Riccati-type differential or integral equation (for finite dimensional systems, see, e.g., the work of Brockett (1970); for infinite dimensional systems consult, e.g., Curtain (1984)). A slightly more general treatment of this topic is presented by Porter (1966, Ch. 4). Our approach to this reduction seems to be new.

5. Minimum energy control problem for infinite dimensional discrete-time control systems

Consider a **linear discrete-time control system** defined by the difference equation

$$x_{k+1} = Ax_k + Bu_k, \tag{10}$$

where k runs through the set of non-negative integers. We assume that $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U, X)$, where the **state space** X as well as the **control space** U are real Hilbert spaces. Let $x_0 \in X$ be an **initial state** and $u := (u_k)_{k=0}^{\tau-1}$ be a **controlling sequence**, where τ denotes a *fixed* positive integer ("final time"). Then

$$x_\tau = A^\tau x_0 + \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k.$$

For discrete-time systems, we formulate the following **fixed endpoints minimum energy control problem**.[2]

For given $x_0 \in X$, $x_{\text{final}} \in X$, and τ being a fixed positive integer, find a controlling sequence $\widehat{u} := (\widehat{u}_k)_{k=0}^{\tau-1}$ such that

$$x_{\text{final}} = A^\tau x_0 + \sum_{k=0}^{\tau-1} A^{\tau-k-1} B \widehat{u}_k \tag{11a}$$

and

$$\Big(\sum_{k=0}^{\tau} \|\widehat{u}_k\|^2\Big)^{1/2} \le \Big(\sum_{k=0}^{\tau} \|u_k\|^2\Big)^{1/2}, \tag{11b}$$

for any controlling sequence $u = (u_k)_{k=0}^{\tau-1}$ satisfying

$$x_{\text{final}} = A^\tau x_0 + \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k. \tag{11c}$$

In order to reformulate the fixed endpoints minimum energy control problem defined by (11) as an extended minimum norm problem discussed in Section 3, we put $\mathcal{H}_u := \ell^2_\tau(U)$ so that the norm of $u \in \mathcal{H}_u$ will be $|u|_2$. We also assume that $\mathcal{H}_0 := X, \mathcal{H}_v := X, \mathcal{H}_z := X \oplus X$. Let

$$R_0 := -A^\tau, \tag{12}$$

$$S := \left[A^{\tau-1}B, A^{\tau-2}B, \dots, AB, B\right]. \tag{13}$$

Let us note that $R_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_v)$, $S \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_v)$, and

$$Su = \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k,$$

for any $u = (u_k)_{k=0}^{\tau-1} \in \mathcal{H}_u = \ell^2_\tau(U)$. Of course, the operators R_0 and S depend on τ. The image of S is known as the **$\boldsymbol{\tau}$-controllable subspace**.

It is clear that the discussed fixed endpoints minimum energy control problem for the system (10) takes the following form.

For given $x_0 \in \mathcal{H}_0 = X$ and $x_{\text{final}} \in \mathcal{H}_v = X$, find (if it is possible) $\widehat{u} = (\widehat{u}_k)_{k=0}^{\tau-1} \in \mathcal{H}_u = \ell^2_\tau(U)$ such that

$$S\widehat{u} = R_0 x_0 + x_{\text{final}},$$

and $|\widehat{u}|_2$ is not greater than the norm $|u|_2$, for any $u = (u_k)_{k=0}^{\tau-1} \in \mathcal{H}_u$ satisfying $Su = R_0 x_0 + x_{\text{final}}$, with R_0 and S defined by (12) and (13), respectively.

There is no doubt that one can employ the results of Section 3 when studying the fixed endpoints minimum energy control problem for the system (10). To this end, let us note that, for the discrete-time system considered, the space $\mathcal{Z} = R^{-1}(\operatorname{Im} S)$ (as defined in Section 3) is as follows:

$$\begin{aligned}\mathcal{Z} &= \big\{(x_0, x_{\text{final}}) \in X \oplus X \mid \exists\, u = (u_k)_{k=0}^{\tau-1} \in \ell^2_\tau(U) : \\ &\qquad x_{\text{final}} = A^\tau x_0 + \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k\big\}.\end{aligned} \tag{14}$$

This space depends on τ.

Let us observe that the minimum energy control problem specified by the relations (11) is well defined if and only if $(x_0, x_{\text{final}}) \in \mathcal{Z}$, with $\mathcal{Z}$ given by (14). Let K (see Proposition 1) denote the linear mapping which maps $(x_0, x_{\text{final}}) \in \mathcal{Z}$ to $\widehat{u}(x_0, x_{\text{final}}) \in \mathcal{H}_u = \ell^2_\tau(U)$, the (unique) solution to the fixed endpoints minimum energy

[2] In view of our results of Section 4 there is no need to consider explicitly a more general linear quadratic problem.

problem considered.

The following theorem is a direct consequence of Theorem 1 and Proposition 5.

Theorem 5. *Consider the fixed endpoints minimum energy control problem specified by the relations (11), and the linear mapping* $K : (x_0, x_{\text{final}}) \mapsto \widehat{u}(x_0, x_{\text{final}})$. *Then* K *is continuous if and only if the* τ*-controllable subspace* $\operatorname{Im} S$ *is closed.*

Let us recall (see, e.g., Fuhrmann, 1972) that a linear discrete-time system is said to be **exactly controllable in τ steps** if for any $x_{\text{final}} \in X$ one can find a controlling sequence $u = (u_k)_{k=0}^{\tau-1}$ such that

$$x_{\text{final}} = \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k,$$

so that when $x_0 = 0$, $x_{\text{final}} = x_\tau$, for some u. In other words, the discussed discrete-time *system is exactly controllable in* τ *steps if and only if* $\operatorname{Im} S = X$.

Corollary 3. *The domain of* K *is equal to* $X \oplus X$ *if and only if the system (10) is* τ*-exactly controllable. Then* K *is continuous.*

Proof. In view of Corollary 2 and Theorem 5, it is sufficient to observe that the space $\mathcal{Z}$ (see (14)) coincides with $X \oplus X$ if and only the τ-controllable subspace is equal to X. ■

The assumption that a system is exactly controllable (or that its τ-controllable subspace is closed) may be too demanding for some infinite dimensional control systems. One can relax this assumption using Theorem 2 of Section 3. To formulate some results in this direction, we introduce below two additional concepts of controllability; they are weaker than that of exact controllability. These concepts are well known (see, e.g., Fuhrmann, 1972; Curtain and Zwart, 1995).

We say that the system (10) is **approximately controllable in τ steps** if for each $x_{\text{final}} \in X$ and any $\varepsilon > 0$ there exists a controlling sequence $u = (u_k)_{k=0}^{\tau-1}$ such that

$$\|x_{\text{final}} - \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k\| \le \varepsilon,$$

so that when $x_0 = 0$ the norm $\|x_{\text{final}} - x_\tau\|$ does not exceed ε, for some u. This means that the discussed *system is approximately controllable in* τ *steps if and only if its* τ*-controllable subspace is dense in* X.

We also need the concept of null-controllability. It is said that the the system (10) is **null-controllable in τ steps** if for every $x_0 \in X$ there exists a controlling sequence $u = (u_k)_{k=0}^{\tau-1}$ such that

$$A^\tau x_0 + \sum_{k=0}^{\tau-1} A^{\tau-k-1} B u_k = 0,$$

so that for each x_0 one can find u steering x_0 to the origin. In other words, the discussed *system is null-controllable in* τ *steps if and only if* $\operatorname{Im} R_0 \subset \operatorname{Im} S$, i.e.,

$$\operatorname{Im} A^\tau \subset \left[A^{\tau-1}B, A^{\tau-2}B, \ldots, AB, B\right].$$

Let (as usual) K denote the linear mapping which maps $(x_0, x_{\text{final}}) \in \mathcal{Z}$ to $\widehat{u}(x_0, x_{\text{final}}) \in \mathcal{H}_u = \ell^2_\tau(U)$. Since K is linear, we have

$$\widehat{u}(x_0, x_{\text{final}}) = K(x_0, x_{\text{final}}) = K_0 x_0 + K_{\text{final}} x_{\text{final}}$$

for appropriate linear mappings K, K_0 and K_{final}.

The following result is merely a rephrasing of Theorem 2.

Theorem 6. *Consider the fixed endpoints minimum energy control problem specified by the relations (11). Assume that the system considered is null-controllable in* τ*-steps, and that its* τ*-controllable subspace (i.e.,* $\operatorname{Im} S$*) is not closed. Let* K_0 *and* K_{final} *be as above. Then* K_0 *is continuous, i.e.,* $K_0 \in \mathcal{L}\big(X, \ell^2_\tau(U)\big)$, *and* $K_{\text{final}} : \operatorname{Im} S \to \ell^2_\tau(U)\big)$ *is linear but discontinuous.*

We also have the following.

Corollary 4. *Assume that the system (10) is null-controllable in* τ*-steps. Let the system be approximately controllable in* τ *steps, but not exactly controllable. Then the conclusion of Theorem 6 is valid, i.e.,* K_0 *is continuous and* K_{final} *is discontinuous.*

6. Minimum energy control problem for infinite dimensional continuous-time control systems

We will consider continuous-time systems. In what follows, we denote by T a *fixed* positive real number. Let a **linear continuous-time control system** be described by the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{15}$$

where t runs through the set of non-negative real numbers. We assume that A is is the infinitesimal generator of a strongly continuous semigroup of continuous linear operators $\big(\Phi(t)\big)_{t\ge 0}$, $B \in \mathcal{L}(U, X)$, where the **state space** X as well as the **control space** U are real Hilbert spaces. We write $L^2((0,T);U)$ for the Hilbert space of all (equivalent classes of) square-integrable functions $[0,T] \to U$, normed in the usual way. Let $x_0 \in X$ be an **initial state** and $u(\cdot) \in L^2((0,T);U)$ be a **controlling**

function. Then we say that

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)Bu(s)\,\mathrm{d}s \tag{16}$$

is a *mild solution of Eqn.(15) on* $[0,T]$. The above formula makes sense for all $x_0 \in X$ and $u(\cdot) \in L^2((0,T);U)$, and it can be shown that $x(\cdot) \in L^2((0,T);X)$. At this point we refer the reader to the works of Balakrishnan (1981) or Curtain and Pritchard (1978) for details and a very clear exposition of various properties of mild (and weak) solutions of differential equations.

For continuous-time systems, we will consider the following **fixed endpoints minimum energy control problem**.[3]

For given $x_0 \in X$, $x_{\text{final}} \in X$, and T being a fixed positive real number, find a controlling function $\widehat{u}(\cdot) \in L^2((0,T);U)$ such that

$$x_{\text{final}} = \Phi(t)x_0 + \int_0^T \Phi(T-s)B\widehat{u}(s)\,\mathrm{d}s \tag{17a}$$

and

$$\left(\int_0^T \|\widehat{u}(s)\|\,\mathrm{d}s\right)^{1/2} \le \left(\int_0^T \|u(s)\|\,\mathrm{d}s\right)^{1/2}, \tag{17b}$$

for any controlling function $u(\cdot)$ satisfying

$$x_{\text{final}} = \Phi(t)x_0 + \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s. \tag{17c}$$

Like in the case of the problem (11), the above fixed endpoint minimum energy control problem can be rewritten as an extended minimum norm problem of Section 3. To this end, it is sufficient to set $\mathcal{H}_u := L^2((0,T);U)$, $\mathcal{H}_0 := X$, $\mathcal{H}_v := X$, $\mathcal{H}_z := X \oplus X$. Let

$$R_0 := -\Phi(T)x_0, \tag{18}$$

$$Su(\cdot) := \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s, \tag{19}$$

for any $u(\cdot) \in L^2((0,T);U)$. Then, for the continuous-time system considered, the space $\mathcal{Z} = R^{-1}(\operatorname{Im} S)$ (as defined in Section 3) is as follows:

$$\mathcal{Z} = \big\{(x_0, x_{\text{final}}) \in X \oplus X \mid \exists u(\cdot) \in L^2((0,T);U) : x_{\text{final}} = \Phi(t)x_0 + \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s\big\}. \tag{20}$$

[3] Of course, we know that there is no need to consider a more general linear quadratic problem.

Let us note that $R_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_v)$ and $S \in \mathcal{L}(\mathcal{H}_u, \mathcal{H}_v)$. In this section we assume that R_0, S and $\mathcal{Z}$ are given by the formulas (18), (19) and (20), respectively. It is clear that the operators R_0, S, and the space $\mathcal{Z}$ depend on T.

The image of the above defined operator S is named the ***T*-controllable subspace**. For a broad class of *infinite dimensional continuous-time systems*, the T-controllable subspace (i.e., $\operatorname{Im} S$) cannot be closed, and therefore $\operatorname{Im} S$ is a proper subspace of X. This takes place when B is compact, or $\Phi(\cdot)$ is a compact semigroup. Then the operator S is compact and has (usually) infinite dimensional image. This important fact is well known (see Balakrishnan, 1981; Curtain and Pritchard, 1978; Kobayashi, 1978; Triggiani, 1975a).

In a similar manner like for discrete-time systems, one can define (see, e.g., Curtain and Pritchard, 1978; Curtain and Zwart, 1995) the concepts of exact controllability, approximate controllability, and null-controllability for a continuous-time system.

Let us recall that a linear continuous-time system is **exactly controllable on $[0,T]$** if for every $x_{\text{final}} \in X$ one can find a controlling function $u(\cdot) \in L^2((0,T);U)$, such that

$$x_{\text{final}} = \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s,$$

so that when $x_0 = 0$, $x_{\text{final}} = x(T)$, for some $u(\cdot)$. In other words, the discussed continuous-time *system is exactly controllable on* $[0,T]$ *if and only if* $\operatorname{Im} S = X$.

The system (15) is said to be **approximately controllable on $[0,T]$** if for each $x_{\text{final}} \in X$ and any $\varepsilon > 0$ there exists a controlling function $u(\cdot) \in L^2((0,T);U)$ such that

$$\|x_{\text{final}} - \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s\| \le \varepsilon,$$

so that when $x_0 = 0$, the norm $\|x_{\text{final}} - x(T)\|$ does not exceed ε for some $u(\cdot)$. This means that the discussed *system is approximately controllable on* $[0,T]$ *if and only if its T-controllable subspace is dense in X.*

The important concept of null-controllability for continuous-time systems is defined as follows. We say that the system (15) is **null-controllable on $[0,T]$** if for every $x_0 \in X$ there exists a controlling function $u(\cdot) \in L^2((0,T);U)$ such that

$$\Phi(t)x_0 + \int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s = 0,$$

so that, for each x_0 one can find $u(\cdot)$ steering x_0 to the origin. In other words, the discussed *system is null-controllable on* $[0,T]$ *if and only if* $\operatorname{Im} R_0 \subset \operatorname{Im} S$.

Various important results concerning the above concepts of controllability have been obtained by Triggiani (1975a; 1975b; 1976).

We know that the minimum energy control problem described by the relations (17) is well defined if and only if $(x_0, x_{\mathsf{final}}) \in \mathcal{Z}$, with $\mathcal{Z}$ given by (20). Then (see Proposition 1) there exists a linear mapping K which maps each $(x_0, x_{\mathsf{final}}) \in \mathcal{Z}$ to $\widehat{u}(x_0, x_{\mathsf{final}}) \in \mathcal{H}_u = L^2((0,T);U)$, the (unique) solution to the fixed endpoints minimum energy problem considered, so that $\widehat{u}(x_0, x_{\mathsf{final}}) = K_0 x_0 + K_{\mathsf{final}} x_{\mathsf{final}}$, for suitable linear mappings. It is obvious that the results analogous to those obtained for our discrete-time problem (11) remain true, *mutatis mutandis*, for the continuous-time fixed endpoints minimum energy problem defined by the relations (17). We record only the following result.

Proposition 8. *Consider the fixed endpoints minimum energy control problem given by the relations (17). Assume that the system (15) is null-controllable on* $[0,T]$. *Let the system be approximately controllable on* $[0,T]$, *but not exactly controllable on* $[0,T]$. *Let* K_0 *and* K_{final} *be defined as usual, so that the optimal solution* $\widehat{u}$ *to (17) can be written as* $\widehat{u}(x_0, x_{\mathsf{final}}) = K_0 x_0 + K_{\mathsf{final}} x_{\mathsf{final}}$. *Then* K_0 *is continuous, i.e.,* $K_0 \in \mathcal{L}\big(X, L^2((0,T);U)\big)$, *and* $K_{\mathsf{final}} : \mathsf{Im}\, S \to L^2((0,T);U)$ *is linear but discontinuous.*

We end this section with the following example of a distributed parameter system.

Example 1. We consider, for $t \in [0,T]$ and $\xi \in [0,1]$, the (one-dimensional) *heat equation*

$$\frac{\partial \theta}{\partial t}(\xi,t) = \frac{\partial^2 \theta}{\partial \xi^2}(\xi,t) + h(\xi,t), \tag{21a}$$

subject to the *boundary condition*

$$\frac{\partial \theta}{\partial \xi}(0,t) = \frac{\partial \theta}{\partial \xi}(1,t) = 0. \tag{21b}$$

Here $\theta(\xi,t)$ denotes the temperature at time t at position ξ. Then the relations (21) describe a (thin homogeneous) metal rod of length one, with (perfectly) insulated endpoints, with some additional heat source that can increase (or decrease) the temperature at each point ξ along the rod, at a given rate $h(\xi,t)$, also known as the *heat source density*.

Our aim it to find a heat source density h such that *the initial temperature distribution* $\theta(\xi,0)$ *will be changed to a given (desired) temperature distribution* $\theta(\xi,T)$, *at time* T, *and the energy used for this, i.e.,*

$$\int_0^T \int_0^1 \big(h(\xi,t)\big)^2 \, \mathrm{d}\xi \, \mathrm{d}t, \tag{22}$$

will be as low as possible.

It is well known (see, e.g., Balakrishnan, 1981; Curtain and Zwart, 1995) that Eqns. (21) can be rewritten as a differential equation of the form (15), with suitable A and B. For this, let $X = U = L^2((0,1);\mathbb{R})$. Let $x(t) := \theta(\cdot,t)$ and $u(t) := h(\cdot,t)$, so that (for each $t \in [0,T]$), $x(t)$ and $u(t)$ are real-valued functions of the (spatial) variable $\xi \in [0,1]$. Observe that

$$x(0) = \theta(\cdot,0) \quad \text{and} \quad x(T) = \theta(\cdot,T)$$

represent the initial temperature distribution and its desired (final) distribution at $t = T$, respectively. For that reason, $x(0)$ *will play the role of* x_0, *and* $x(T)$ *will be our* x_{final}; see the relations (17).

The left-hand side of Eqn. (21a) can be identified with $\dot{x}(t)$, the derivative of x with respect to t. The second term of the right-hand side of Eqn. (21a) can be represented by $u(t)$. It follows that, when expressing the relations (21) as a differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, we should assume that $B = I$, the identity operator $U \to X$ $(= U)$.

To describe the operator A, let us consider any $x \in X$. Such x is a function of the spatial variable $\xi \in [0,1]$. The right-hand side of (21a) contains the term $(\partial^2\theta/\partial\xi^2)(\xi,t)$, i.e., the second derivative of x with respect to ξ. It follows that A is an ordinary second order differential operator, i.e., the operator defined by the formula

$$Ax = \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}.$$

The domain $\mathsf{dom}\, A$ of A should reflect differentiability conditions, and also the boundary condition imposed by (21b). It is known (and not very difficult to check) that the appropriate domain of A coincides with the linear subspace of $X = L^2((0,1);\mathbb{R})$ containing all absolutely continuous functions x of the (spatial) variable ξ, whose first derivative (with respect to ξ) is absolutely continuous and the second derivative belongs to $L^2((0,1);\mathbb{R})$, and such that the boundary condition (21b) is satisfied, i.e., $(\mathrm{d}x/\mathrm{d}\xi)(0) = (\mathrm{d}x/\mathrm{d}\xi)(1) = 0$. One can check that the above described linear operator $A : \mathsf{dom}\, A \to X$ is the infinitesimal generator of a strongly continuous semigroup. Moreover, A belongs to the class of Riesz-spectral operators, and the semigroup $\big(\Phi(t)\big)_{t \geq 0}$ generated by A can be written in an explicit form. For details, the interested reader should consult Theorem 2.3.5 and Examples 2.1.1, 2.3.7 in the work of Curtain and Zwart (1995).

We see that the discussed heat equation (21) can be represented as a linear continuous-time control system described by a differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, with X, U and A, B described above. Therefore one can reformulate the problem of minimizing energy (22) as a fixed endpoints minimum energy control problem (17). Then $\mathcal{H}_u = L^2\big((0,T); L^2((0,1);\mathbb{R})\big)$. Since $u(t) := h(\cdot,t))$, for any $u \in \mathcal{H}_u$, we have

$$\|u\|^2 = \int_0^T \int_0^1 \big(h(\xi,t)\big)^2 \,\mathrm{d}\xi\,\mathrm{d}t,$$

the norm $\|u\|$ of u being evaluated in $\mathcal{H}_u$. Hence, the problem of minimizing energy (22) falls into the framework we know from the beginning of this section.

It remains to check whether or not the linear continuous-time control system $\dot{x}(t) = Ax(t) + Bu(t)$ representing the heat equation (21) is exactly controllable, approximately controllable, or null-controllable. It happens that (for arbitrary positive T) the discussed continuous-time *system is approximately controllable on* $[0,T]$*, null-controllable on* $[0,T]$*, but never exactly controllable*. These facts are well known, and can be justified with the aid of various arguments. The simplest way to prove them is to use the controllability criteria presented by Curtain and Zwart (1995, Chap. 4). It has been done in the existing literature. In particular, Example 4.1.10 of Curtain and Zwart (1995) proves that this system is never exactly controllable on $[0,T]$, but it is null-controllable. To prove that this system is approximately controllable on $[0,T]$, one can use the duality between observation and control. Example 4.1.15 of Curtain and Zwart (1995) contains all necessary details.

Now, one can use our Proposition 8. Since we know that the heat equation considered is approximately controllable, null-controllable, but never exactly controllable, we conclude that the *solution to the minimum norm problem for the system (21) will depend continuously on the initial state* $x(0) = \theta(\cdot,t)$*, but it cannot continuously depend on the final condition* $x(T) = \theta(\cdot,T)$.

References

Athans, M. (1971). The role and use of the stochastic linear-quadratic-Gaussian problem in control system design, *IEEE Transactions on Automatic Control*, **AC-16**(6): 529–552.

Aubin, J.-P. (2000). *Applied Functional Analysis*, Wiley, New York, NY.

Balakrishnan, A.V. (1981). *Applied Functional Analysis*, Springer, New York, NY.

Brockett, R.F. (1970). *Finite-Dimensional Linear Systems*, Wiley, New York, NY.

Chow, G.P. (1976). *Analysis and Control of Dynamic Economic Systems*, Wiley, New York, NY.

Corless, M.J. and Frazho, A.E. (2003). *Linear Systems and Control. An Operator Perspective*, Marcel Dekker, New York, NY.

Curtain, R.F. (1984). Linear-quadratic control problem with fixed endpoints in infinite dimensions, *Journal of Optimization Theory and Its Applications* **44**(1): 55–74.

Curtain, R.F. and Pritchard A.J. (1978). *Infinite-Dimensional Linear Systems Theory*, Springer, Berlin.

Curtain, R.F. and Zwart H. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer, New York, NY.

Douglas, R.G. (1966). On majorization, factorization, and range inclusion of operators on Hilbert space, *Proceedings of the American Mathematical Society* **18**(2): 413–415.

Emirsajłow, Z. (1989). Feedback control in LQCP with a terminal inequality constraint, *Journal of Optimization Theory and Applications* **62**(3): 387–403.

Evans, L.C. (2010). *Partial Differential Equations*, American Mathematical Society, Providence RI.

Federico, S. (2011). A stochastic control problem with delay arising in a pension fund model, *Finance and Stochastics* **15**(3): 421–459.

Fuhrmann, P.A. (1972). On weak and strong reachability and controllability of infinite-dimensional linear systems, *Journal of Optimization Theory and Its Applications* **9**(2): 77–89.

Kandilakis, D. and Papageorgiou, N.S. (1992). Evolution inclusions of the subdifferential type depending on a parameter, *Commentationes Mathematicae Universitatis Carolinae* **33**(3): 437–449.

Kendrick, D.A. (1981). *Stochastic Control for Economic Models*, McGraw-Hill, New York, NY.

Kobayashi, T. (1978). Some remarks on controllability for distributed parameter systems, *SIAM Journal on Control and Optimization* **16**(5): 733–742.

Laurent, P.-J. (1972). *Approximation et Optimisation*, Hermann, Paris.

Ljungqvist, L. and Sargent, T.J. (2004). *Recursive Macroeconomic Theory*, MIT Press, Cambridge, MA.

Luenberger, D.G. (1969). *Optimization by Vector Space Methods*, Wiley, New York, NY.

Papageorgiou, N.S. (1991). On the dependence of the solutions and optimal solutions of control problems on the control constraint set, *Journal of Mathematical Analysis and Applications* **158**(2): 427–447.

Porter, W.A. (1966). *Modern Foundations of System Engineering*, Macmillan, New York, NY.

Przyłuski, K.M. (1981). Remarks on continuous dependence of an optimal control on parameters, *in* O. Moeschlin and D. Pallaschke (Eds.), *Game Theory and Mathematical Economics*, North-Holland, Amsterdam, pp. 333–337.

Rolewicz, S. (1987). *Functional Analysis and Control Theory. Linear Systems*, PWN, Warsaw, (in Polish).

Sent, E.-M. (1998). *The Evolving Rationality of Rational Expectations: An Assessment of Thomas Sargent's Achievements*, Cambridge University Press, Cambridge.

Triggiani, R. (1975a). A note on the lack of exact controllability for mild solutions in Banach spaces, *SIAM Journal on Control and Optimization* **15**(3): 407–411.

Triggiani, R. (1975b). On the lack of exact controllability for mild solutions in Banach spaces, *Journal of Mathematical Analysis and Applications* **50**(2): 438–446.

Triggiani, R. (1976). Extensions of rank conditions for controllability and observability to Banach spaces and unbounded operators, *SIAM Journal on Control and Optimization* **14**(2): 313–338.

K. Maciej Przyłuski studied electronic engineering at the Warsaw University of Technology, Poland. He received the M.Sc. and Ph.D. degrees (in control sciences) from that university in 1974 and 1977, respectively. He was an assistant professor at the Department of Electronic Engineering of the Warsaw University of Technology from 1978 to 1979. From 1980 to 2003 he worked at the Institute of Mathematics, Polish Academy of Sciences, as an associate professor. In 1990 he received the D.Sc. degree from that institute. From 1999 to 2000 he was also employed at the Systems Research Institute, Polish Academy of Sciences. Since the year 2000 he has been lecturing at various private universities in Poland. Currently, he is a professor of Collegium Mazovia, Siedlce, Poland. He has held visiting positions at Universität Bremen (Germany), Vrije Universiteit (Amsterdam, the Netherlands), Technische Hogeschool Eindhoven (the Netherlands), Universität Karlsruhe (Germany), Technische Universität Graz (Austria), the University of Texas at Arlington (USA), the Georgia Institute of Technology (Atlanta, USA), as well as a number of short visiting appointments. His research interests are in all aspects of linear systems and control theory, and also in their applications. His present research interests include also microeconomics and decision theory. He has contributed to the theory of distributed parameter systems, functional-differential equations, time-varying systems, stability and stabilizability theory, linear sequential machines, observability of systems with disturbances, and implicit (singular) control systems. He is the author of about 50 papers and a co-author of a monograph on linear finite-dimensional control systems.

Received: 30 September 2013
Revised: 7 May 2014