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POSITIVITY OF FRACTIONAL DESCRIPTOR LINEAR DISCRETE–TIME SYSTEMS

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The positivity of fractional descriptor linear discrete-time systems is investigated. The solution to the state equation of the systems is derived. Necessary and sufficient conditions for the positivity of fractional descriptor linear discrete-time systems are established. The discussion is illustrated with numerical examples.

Keywords: fractional, descriptor, linear, discrete-time, system, stability, solution, positivity.

1. Introduction

A dynamical system is called positive if its state variables take nonnegative values for all nonnegative inputs and nonnegative initial conditions. Positive linear systems were investigated by Berman and Plemmons (1994), Farina and Rinaldi (2000) or Kaczorek (2002), who also studied positive nonlinear systems (Kaczorek, 2016; 2015a; 2014; 2015b; 2015c).

Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Linear systems with different fractional orders were addressed by Busłowicz (2012), Kaczorek (2010; 2011a) and Sajewski (2016). Descriptor (singular) linear systems were analyzed by Borawski (2018), Kaczorek (2014; 2016b; 2019; 2012; 1997; 1993) or Ali and Diego (2012), and the stability of a class of nonlinear fractional-order systems was studied by Kaczorek (2016a; 2011b) or Xiang-Jun *et al.* (2008). Fractional positive continuous-time linear systems and their reachability were addressed by Kaczorek (2008). Application of the Drazin inverse to the analysis of descriptor fractional discrete-time linear systems was presented by Kaczorek (2013), while the stability of discrete-time switched systems with unstable subsystems was studied by Zhang *et al.* (2014a). Robust stabilization of discrete-time positive switched systems with uncertainties was addressed by Zhang *et al.* (2014b). A comparison of three methods of analysis of descriptor fractional systems was presented by Sajewski (2016a). The stability of linear fractional order systems with delays was analyzed by Busłowicz (2008), and simple conditions for practical stability of positive fractional systems were proposed by Busłowicz and Kaczorek (2009). The stability of interval positive continuous-time linear systems was addressed by Kaczorek (2018). Positive controllability of positive dynamical systems was considered by Klamka (2002), while some remarks on stability of positive linear systems were given by Mitkowski (2000), along with dynamical properties of Metzler systems (Mitkowski, 2008).

In this paper the positivity of fractional descriptor discrete-time linear systems will be investigated. The paper is organized as follows. In Section 2 basic definitions of the Drazin inverse of matrices are recalled and the solution to the system state equation is derived. Necessary and sufficient conditions for the positivity of fractional descriptor linear discrete-time systems are established in Section 3. Concluding remarks are given in Section 4.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices; $\mathbb{R}_+^{n\times m}$, the set of $n\times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$; I_n, the $n \times n$ identity matrix.

2. Fractional descriptor linear discrete-time systems

Consider the fractional descriptor linear system

$$E\Delta^{\alpha} x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = 0,1,2,\dots, \quad 0<\alpha<1, \tag{1a}$$

$$y_i = Cx_i, \tag{1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are respectively the state, input and output vectors, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and

$$\Delta^{\alpha} x_i = \sum_{j=0}^{i} (-1)^j \binom{\alpha}{j} x_{i-j}, \tag{1c}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j=1,2,\dots \end{cases}$$

is the fractional α-order difference of x_i.

It is assumed that

$$\det[E\lambda - A] \neq 0 \quad \text{for some } s \in \mathbb{C}. \tag{2}$$

Definition 1. For any matrix $\bar{E} = [E\lambda - A]^{-1}E \in \mathbb{R}^{n\times n}$ there exists a unique *Drazin inverse* $\bar{E}^D \in \mathbb{R}^{n\times n}$ defined by the conditions

$$\bar{E}^D\bar{E} = \bar{E}\bar{E}^D, \tag{3a}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{3b}$$

$$\bar{E}^D\bar{E}^{\mu+1} = \bar{E}^{\mu}, \tag{3c}$$

where μ is the smallest nonnegative integer such that

$$\operatorname{rank}\bar{E}^{\mu} = \operatorname{rank}\bar{E}^{\mu+1}, \tag{3d}$$

and λ is chosen so that (2) is satisfied.

It is easy to check that for the matrices

$$P = \bar{E}^D, \quad \bar{E} = \bar{E}^D[E\lambda - A]^{-1}E, \quad \hat{A} = \bar{E}^D[E\lambda - A]^{-1}A \tag{4}$$

the following relations hold:

$$P^k = P \quad \text{for} \quad k = 2,3,\dots, \tag{5a}$$

$$P\hat{A} = \hat{A}P = \hat{A}. \tag{5b}$$

Premultiplying (1a) by the matrix $\bar{E}^D[E\lambda - A]^{-1}$, we obtain

$$P\Delta^{\alpha}x_{i+1} = \hat{A}x_i + \hat{B}u_i, \tag{6a}$$

where

$$\hat{B} = \bar{E}^D[E\lambda - A]^{-1}B. \tag{6b}$$

Substituting (1c) into (6a), we obtain

$$Px_{i+1} = \hat{A}_{\alpha}x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + \hat{B}u_i, \quad i \in \mathbb{Z}_+, \tag{7a}$$

where

$$\hat{A}_{\alpha} = \hat{A} + P\alpha, \tag{7b}$$

$$c_j = (-1)^{j+1}\binom{\alpha}{j}, \quad j = 1,2,\dots. \tag{7c}$$

From (7a) it follows that the fractional system is equivalent to the descriptor system with an increasing number of delays.

Theorem 1. *The solution of Eqn. (7a) has the form*

$$x_i = \Phi_i x_0 + \sum_{j=0}^{i-1} \Phi_{i-j-1}\hat{B}u_j, \quad i \in \mathbb{Z}_+,$$

$$x_0 = \operatorname{Im} P = Pd, \quad d \in \mathbb{R}^n : \textit{arbitrary}, \tag{8a}$$

where the matrix Φ_j *is given by*

$$\Phi_{j+1} = \hat{A}_{\alpha}\Phi_j + \sum_{k=2}^{j+1} c_k \Phi_{j-k+1}, \Phi_0 = I_n, \tag{8b}$$

and $\hat{A}_{\alpha}$, c_k *are defined by (7b) and (7c), respectively.*

Proof. Using (7) and (8), it is easy to verify that

$$\begin{aligned} Px_{i+1} &= P\left[\Phi_{i+1}x_0 + \sum_{j=0}^{i} \Phi_{i-j}\hat{B}u_j\right] \\ &= P\hat{A}_{\alpha}\Phi_i x_0 + \sum_{j=0}^{i} P\Phi_{i-j}\hat{B}u_j \\ &\quad + \sum_{k=2}^{i+1} c_k\Phi_{i-k+1}x_0 \\ &= \hat{A}_{\alpha}\left[\Phi_i x_0 + \sum_{j=0}^{i-1}\Phi_{i-j-1}\hat{B}u_j\right] \\ &\quad + \sum_{j=2}^{i+1} c_j x_{i-j+1} + \hat{B}u_i, \end{aligned} \tag{9}$$

since, by (5b), $P\hat{A}_{\alpha} = \hat{A}_{\alpha}$. Therefore, the solution (8) satisfies Eqn. (7a). ■

Example 1. Consider the fractional descriptor system (1a) with

$$E = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u(t) = \begin{cases} 1 & \text{for} \quad i \in \mathbb{Z}_+, \\ 0 & \text{for} \quad i < 0. \end{cases} \tag{10}$$

The system satisfies the assumption (2) since

$$\det[E\lambda - A] = \begin{vmatrix} 0 & -1 \\ -\lambda & \lambda \end{vmatrix} = -\lambda. \tag{11}$$

Choosing $\lambda = -1$ and using (10), we obtain

$$\begin{aligned} \bar{E} &= [E\lambda - A]^{-1}E = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} &= [E\lambda - A]^{-1}A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{12}$$

In this case, the Drazin inverse matrix has the form

$$\bar{E}^D = \bar{E} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \tag{13}$$

and

$$\begin{aligned} \hat{A} &= \bar{E}^D\bar{A} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{B} &= \bar{E}^D[E\lambda - A]^{-1}B \\ &= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \end{aligned} \tag{14a}$$

$$\hat{A}_\alpha = \hat{A} + P\alpha = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix}. \tag{14b}$$

Note that, in this case,

$$P\hat{A}_\alpha = P^2\alpha = P\alpha = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix}. \tag{15}$$

Using (8) and (14), we obtain

$$\begin{aligned} \Phi_1 &= \hat{A}_\alpha = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix}, \\ \Phi_2 &= \hat{A}_\alpha^2 + c_2 I_2 = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix}^2 \\ &\quad + \frac{\alpha(1-\alpha)}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \Phi_3 &= \hat{A}_\alpha^3 + c_2\hat{A}_\alpha + c_3 I_2 = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix}^3 \\ &\quad + \frac{\alpha(1-\alpha)}{2}\begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix} \\ &\quad + \frac{\alpha(1-\alpha)(2-\alpha)}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{16a}$$

and

$$\begin{aligned} x_1 &= \Phi_1 x_0 + \hat{B}u_0, \\ x_2 &= \Phi_2 x_0 + \Phi_1\hat{B}u_0 + \hat{B}u_1, \\ x_3 &= \Phi_3 x_0 + \Phi_2\hat{B}u_0 + \Phi_1\hat{B}u_1 + \hat{B}u_2, \end{aligned} \tag{16b}$$

where

$$x_0 = \operatorname{Im} P = \operatorname{Im}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

(the set of vectors $\left[\begin{smallmatrix} x_{10} \\ 0 \end{smallmatrix}\right]$, where x_{10} is arbitrary). ♦

Theorem 2. *The solution x_i of Eqn. (1a) satisfies*

$$Px_i = x_i, \quad i \in \mathbb{Z}_+, \tag{17}$$

that is, the solution x_i starting from x_0 in the subspace $\operatorname{Im} P$ remains in this subspace for all $i \in \mathbb{Z}_+$.

Proof. From (8), we have

$$\begin{aligned} Px_i &= P\Phi_i x_0 + \sum_{j=0}^{i-1} P\Phi_{i-j-1}\hat{B}u_j \\ &= \Phi_i x_0 + \sum_{j=0}^{i-1} P\Phi_{i-j-1}\hat{B}u_j = x_i, \end{aligned} \tag{18}$$

since $P\hat{A}_\alpha = \hat{A}_\alpha$ and

$$\begin{aligned} P\Phi_i &= P\hat{A}_\alpha\Phi_{i-1} + \sum_{k=1}^{i} c_k\Phi_{i-k} \\ &= \Phi_i, \quad i = 1, 2, \ldots. \end{aligned} \tag{19}$$

Therefore, the solution x_i, $i \in \mathbb{Z}_+$, starting from $x_0 \in \operatorname{Im} P$, remains in this subspace for all $i \in \mathbb{Z}_+$. ■

Example 2. (*Continuation of Example 1*) In this case

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \tag{20}$$

and the subspace

$$\operatorname{Im} P = Pd = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

consists of all vectors with a zero second component and (15) holds. ♦

3. Positivity of fractional descriptor linear discrete-time systems

The following lemma will be used in our further discussion.

Lemma 1. *For the fractional discrete-time linear system*

$$\Delta^\alpha z_{i+1} = Mz_i, \quad M \in \mathbb{R}^{n\times n}, \quad 0 < \alpha < 1, \tag{21}$$

the implication

$$Fz_0 \in \mathbb{R}_+^p \quad \text{then} \quad Fz_i \in \mathbb{R}_+^p \quad \text{for } F \in \mathbb{R}^{p\times n}, \quad i \in \mathbb{Z}_+ \tag{22}$$

holds true if and only if there exists $H \in \mathbb{R}_+^{p\times p}$ such that

$$FM = HF. \tag{23}$$

Proof. Premultiplying (21) by the matrix F, we obtain

$$\Delta^\alpha F z_{i+1} = FMz_i, \quad i \in \mathbb{Z}_+. \tag{24}$$

Equation (24) has the solution $Fz_i \in \mathbb{R}^p_+$, $i \in \mathbb{Z}_+$ if and only if (23) holds true. Note that the equation

$$\Delta^\alpha F z_{i+1} = HFz_i, \quad i \in \mathbb{Z}_+, \tag{25}$$

has the solution $Fz_i \in \mathbb{R}^p_+$, $i \in Z_+$ if and only if $H \in \mathbb{R}^{p\times p}_+$. ■

First, let us consider the autonomous fractional descriptor discrete-time system

$$E\Delta^\alpha z_{i+1} = Az_i \tag{26}$$

obtained from (1a) for $Bu_i = 0$.

Definition 2. The autonomous fractional descriptor system (26) is called (internally) *positive* if $x_i \in \mathbb{R}^n_+$, $i \in \mathbb{Z}_+$, for any admissible initial conditions $x_0 \in \mathbb{R}^n_+$ ($x_0 \in \operatorname{Im} P$).

Theorem 3. *The fractional descriptor system* (26) *is positive if and only if there exists a matrix* $G \in \mathbb{R}^{n\times n}$ *such that*

$$H = \hat{A}_\alpha + G(I_n - P) \in \mathbb{R}^{n\times n}_+, \tag{27}$$

where $\hat{A}_\alpha$ *and* P *are defined by* (4).

Proof. By Lemma 1, the system (26) is positive if and only if there exists a matrix $H \in \mathbb{R}^{p\times p}_+$ such that

$$\hat{A}_\alpha = HP. \tag{28}$$

The solution of Eqn. (28) is given by (27) since, by (5b) and (5a), $\hat{A}_\alpha P = \hat{A}_\alpha$, $P^2 = P$ and

$$HP = \hat{A}_\alpha P + G(I_n - P)P = \hat{A}_\alpha P = \hat{A}_\alpha. \tag{29}$$

This completes the proof. ■

Note that the system (26) can be positive even though the matrix $\hat{A}_\alpha$ is not nonnegative. If $\hat{A}_\alpha \in \mathbb{R}^{n\times n}_+$, then we have the following result.

Corollary 1. *The fractional descriptor system* (26) *is positive if* $\hat{A}_\alpha \in \mathbb{R}^{n\times n}_+$. *In this case, we may choose in* (27) $G = 0$.

Example 3. (*Continuation of Example 1*) Consider the autonomous fractional system (26) with

$$E = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad 0 < \alpha < 1. \tag{30}$$

This system is positive since by Theorem 3 there exists a matrix $G \in \mathbb{R}^{2\times 2}$ such that the condition (27) is satisfied. For (30) and

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \tag{31}$$

from (27) we obtain

$$\begin{aligned} H &= \hat{A}_\alpha + G(I_n - P) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & -\alpha \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{2\times 2}_+ \end{aligned} \tag{32}$$

for any $\alpha \in (0, 1)$. Note that the matrix $\hat{A}_\alpha$ has one negative entry. ♦

In a general case, the positivity of the fractional descriptor system (1) is defined as follows.

Definition 3. The fractional descriptor system (1) is called (internally) *positive* if $x_i \in \mathbb{R}^n_+$ and $y_i \in \mathbb{R}^p_+$, $i \in \mathbb{Z}_+$, for any admissible initial conditions $x_0 \in \mathbb{R}^n_+$ ($x_0 \in \operatorname{Im} P$) and all $u_i \in \mathbb{R}^m_+$, $i \in \mathbb{Z}_+$.

Theorem 4. *The fractional descriptor system (1) is positive if and only if there exists a matrix* $G \in \mathbb{R}^{n\times n}$ *such that* (27) *holds true and*

$$\hat{B} \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+. \tag{33}$$

Proof. The proof of (27) is the same as that of Theorem 3. Note that

$$\sum_{j=0}^{i-1} \Phi_{i-j-1}\hat{B}u_j \in \mathbb{R}^n_+ \quad \text{for} \quad i \in \mathbb{Z}_+ \tag{34}$$

if and only if $\hat{B} \in \mathbb{R}^{n\times m}_+$ since $u_i \in \mathbb{R}^m_+$, $i \in \mathbb{Z}_+$, is arbitrary. Similarly, $y_i \in \mathbb{R}^p_+$, $i \in \mathbb{Z}_+$ if and only if $C \in \mathbb{R}^{p\times n}_+$ since $x_i \in \mathbb{R}^m_+$, $i \in \mathbb{Z}_+$, can be arbitrary. This completes the proof. ■

Example 4. Consider the system (1) with

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [\ 1 \quad 0\ ]. \tag{35}$$

The assumption (2) is satisfied for $\lambda = 0$ and

$$\begin{aligned} \bar{E} &= [-A]^{-1}E = \begin{bmatrix} -1 & -0.2 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} &= [-A]^{-1}A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned} \tag{36}$$

and

$$\bar{E}^D = \bar{E} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{37}$$

Using (36) and (37), we obtain

$$\begin{aligned} P &= \bar{E}^D \bar{E} = \bar{E} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \\ \hat{A} &= \bar{E}^D \bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \\ \hat{A}_\alpha &= \hat{A} + P\alpha = \begin{bmatrix} \alpha+1 & \alpha+1 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

Note that the condition (27) for the positivity of the system is satisfied for $G = 0$. Therefore, by Theorem 4 the system is positive since the matrices $\hat{B}$ and C defined by (35) satisfy the condition (33). ♦

4. Concluding remarks

The positivity of fractional descriptor linear discrete-time systems was investigated. The solution to the state equation of the fractional descriptor linear discrete-time system was derived (Theorems 1 and 2). Necessary and sufficient conditions for the positivity of fractional descriptor linear discrete-time systems was established (Theorems 3 and 4). The discussion were illustrated with numerical examples.

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