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ON THE CONVERGENCE OF SIGMOIDAL FUZZY GREY COGNITIVE MAPS

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Fuzzy cognitive maps (FCMs) are recurrent neural networks applied for modelling complex systems using weighted causal relations. In FCM-based decision-making, the inference about the modelled system is provided by the behaviour of an iteration. Fuzzy grey cognitive maps (FGCMs) are extensions of fuzzy cognitive maps, applying uncertain weights between the concepts. This uncertainty is expressed by the so-called grey numbers. Similarly as in FCMs, the inference is determined by an iteration process which may converge to an equilibrium point, but limit cycles or chaotic behaviour may also turn up. In this paper, based on the grey connections between the concepts and the parameters of the sigmoid threshold function, we give sufficient conditions for the existence and uniqueness of fixed points of sigmoid FGCMs.

Keywords: fuzzy cognitive map, grey system theory, fuzzy grey cognitive map, fixed point.

1. Introduction

Decision-making problems are often too complex to be solved by classical methods, especially when several uncertain or imprecise factors are present (Carlsson and Fullér, 2011). A large number of successful techniques are based on cognitive or fuzzy models (Papageorgiou and Salmeron, 2014; Bartczuk *et al*., 2016), which are extremely useful when a high number of interrelated factors should be considered by the decision maker and these factors form a complex system (Busemeyer, 2001).

Based on the pioneer work of Axelrod (1976), fuzzy cognitive maps were introduced by Kosko (1986) as a modelling method that can effectively represent causal knowledge and uncertain information of complex systems. The main characteristics of the systems are represented by nodes in a graph, while causality and the strength of the relationship are represented by weighted, directed edges (Felix *et al*., 2017). Moreover, quick simulation of complex models (Stylios and Groumpos, 2004) is also possible. Applications of fuzzy cognitive maps include a large variety of scientific and engineering fields, such as social sciences (Carvalho, 2013), economic problems (Ferreira *et al*., 2017), hydrology (Lorenz *et al*., 2016), waste management (Buruzs *et al*., 2015) and the complex problem of Brexit (Ziv *et al*., 2018), just to mention a few examples. The diversity of these examples verifies the wide range applicability of FCM based modelling in cases where classical methods are not able to provide satisfactory solutions. A review of applications and trends of FCMs based modelling is provided by Papageorgiou and Salmeron (2013).

The nodes of the graph represent specific characteristics or subsystems of the original system, and they are usually called ‘concepts’ in the FCM based approach. The current states of the concepts are represented by numbers from the interval $[0, 1]$ (in some cases the interval $[-1, 1]$ is applied (Tsadiras, 2008)), which are the so-called activation values. Concepts have their initial activation values, but these change quickly during the consecutive steps of the simulation. The limit

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of the iteration process is used in the representation of the modelled system. Although the final conclusion relies on the assumption that the iteration process converges to an equilibrium point (fixed point), limit cycles or chaotic behaviour may also occur. Consequently, it is essential to know whether a certain FCM has a unique fixed point.

Fuzzy grey cognitive maps (FGCMs) are extensions of fuzzy cognitive maps, designed for the case when only imprecise information is provided about the relationships between the specific factors of the system (Salmeron, 2010). Similarly to the classical FCMs, the iteration may arrive at a fixed point or a limit cycle, or show chaotic patterns. Therefore, the problem of fixed points plays a crucial role in the case of FGCMs, too.

The rest of the paper is organized as follows. Section 2 contains a short introduction of the basic notions and behaviour of fuzzy cognitive maps. Section 3 presents the mathematical tools applied in the paper and the notion of fuzzy grey cognitive maps. In Section 4, various sufficient conditions are provided for the existence and uniqueness of fixed points of FGCMs with the log-sigmoid, hyperbolic tangent and arbitrary sigmoid-like threshold functions. In Section 5, we briefly summarize the results.

2. Basic notions of fuzzy cognitive maps

Fuzzy cognitive maps use directed graphs in which constant weights are assigned to the edges from the interval $[-1, 1]$ to express the strength and direction of causal connections. The nodes represent specific factors of the modelled system and are called ‘concepts’ in FCM theory. The current states of the concepts are also characterized by numbers in the $[0, 1]$ or $[-1, 1]$ interval; these are the ‘activation values.’

The system can be formally defined by a quadruple (C, W, A, f), where $C = \{C_1, C_2, \dots, C_n\}$ is the set of n concepts, $W : (C_i, C_j) \to w_{ij} \in [-1, 1]$ is a function which associates a causal value (weight) w_{ij} to each edge connecting the nodes C_i and C_j, describing how strongly concept C_i is influenced by concept C_j. The sign of w_{ij} indicates whether the relationship between C_j and C_i is direct or inverse. Thus the connection or weight matrix $W \in \mathbb{R}^{n\times n}$ gathers the system causality.

The function $A\colon C_i \to A_i(k)$ assigns an activation value $A_i(k) \in \mathbb{R}$ to each node C_i at each time step k ($k = 1, 2, \dots$) during the simulation.

A transformation or threshold function $f : \mathbb{R} \to [0, 1]$ calculates the activation value of concepts and keeps them in the allowed range. The most widely used continuous threshold function is the log-sigmoid (sometimes mentioned simply as sigmoid) function

$$f(x) = \frac{1}{1 + e^{-\lambda x}}. \tag{1}$$

In some cases, the required range is the interval $[-1, 1]$, so a function $f : \mathbb{R} \to [-1, 1]$ is applied, which is usually a hyperbolic tangent function ($f(x) = \tanh(\lambda x)$). In both cases the parameter λ influences the slope of the function: the higher the value of λ, the steeper the transition phase of the function.

The iteration which calculates the values of the concept may or may not include self-feedback. In a general form, it can be written as

$$A_i(k) = f\left(\sum_{j=1, j\neq i}^{n} w_{ij} A_j(k-1) + d_i A_i(k-1)\right), \tag{2}$$

where $A_i(k)$ is the value of concept C_i at discrete time k, w_{ij} is the weight of the connection from concept C_j to concept C_i, and d_i expresses the possible self-feedback. If we include the self-feedback in the weight matrix W, the equation can be rewritten in a simpler form (here w_i denotes the i-th row of W and $A(k-1)$ is the concept vector after $k-1$ iteration steps):

$$A(k) = \left[f\left(w_1 A(k-1)\right), \dots, f\left(w_n A(k-1)\right)\right]^T. \tag{3}$$

Continuous FCMs (FCMs with the continuous threshold function) may behave chaotically, produce limit cycles or converge to a fixed point attractor (Tsadiras, 2008). Chaotic behaviour means that the activation vector never stabilizes. If a limit cycle occurs, a specific number of consecutive state vectors turn up repeatedly. In the case of a fixed point attractor, the state vector stabilizes after a certain number of iterations (Nápoles *et al.*, 2017; 2016).

The behaviour of the iteration depends on the threshold function applied and its parameter(s), the elements (weights) of the extended weight matrix and the topology of the map.

Example 1. Let us consider the following example to demonstrate the behaviour of a fuzzy cognitive map. The topology is shown in Fig. 1. The weights of the connections are stored in matrix W:

$$W = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & -0.3 \\ 0 & 0 & -0.7 & 0 & 0 & 0.1 \\ 0.6 & 0 & 0 & -0.4 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.9 & -0.9 & 0 \end{bmatrix}. \tag{4}$$

The threshold function is the log-sigmoid function with parameter $\lambda = 5$. The vector of the initial activation values is the following:

$$A(0) = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}^T. \tag{5}$$

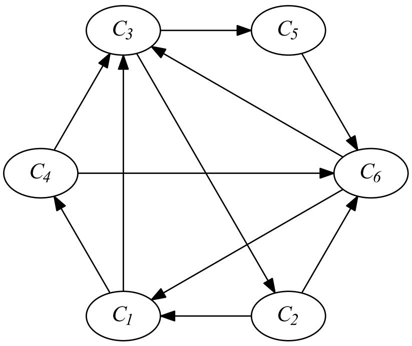

Fig. 1. Graph of the example model. The weight of connection from C_j to C_i is w_{ij}, stored in matrix W.

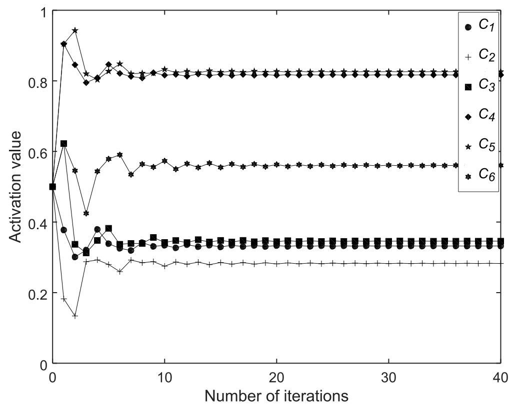

Fig. 2. Activation values of the concepts $C_1, \ldots, C_6$ vs. the number of iterations, $\lambda = 5$.

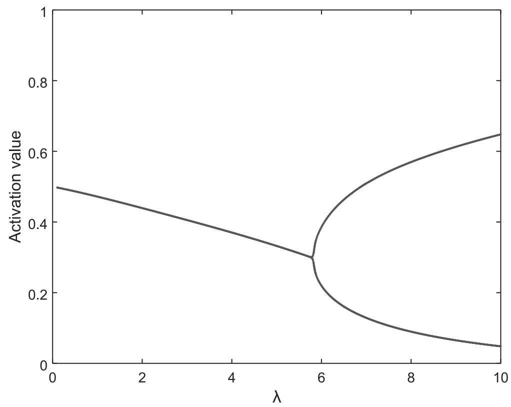

Fig. 3. Activation values of concept C_1 after a large number of iterations vs. parameter λ. Over a certain value of parameter λ, a limit cycle occurs.

As we can observe in Fig. 2, the activation values converge quickly, in a reasonable number of iteration steps. But, if we change the value of λ, the model produces strange behaviour after a certain point (see Fig. 3); namely, it produces a limit cycle (the concept values oscillate between two values), instead of a fixed point. ♦

3. Mathematical background

The problem of the existence and uniqueness of fixed points of sigmoid fuzzy cognitive maps was first discussed by Boutalis *et al*. (2009) for the case when the parameter of the log-sigmoid threshold function is $\lambda = 1$ (so the function was $f(x) = 1/(1 + e^{-x})$). The possible number of fixed points was analysed by Knight *et al*. (2014). The results of Boutalis *et al*. (2009) were generalized by Harmati *et al*. (2018).

3.1. Contraction mapping. Since FCM based decision-making relies on the assumption that the iteration converges to an equilibrium point (fixed point), it is straightforward to examine the question from the viewpoint of the iteration function. Roughly speaking, if the iteration function contracts the space, then it has a unique fixed point. More precisely, we have the following.

Definition 1. (*Contraction*) Let (X, d) be a metric space. A mapping $f\colon X \to X$ is a *contraction mapping* or *contraction* if there exists a constant c (independent from x and y), with $0 \le c < 1$, such that

$$d\left(f(x), f(y)\right) \le cd(x, y). \tag{6}$$

We should note here that the notion of contraction is related to the distance metric d. It may happen that a function is a contraction with respect to one distance metric but not with respect to another distance metric. Let $f\colon X \to X$; then a point $x^* \in X$ such that $f(x^*) = x^*$ is a fixed point of f. The following theorem provides a sufficient condition for the existence and uniqueness of a fixed point.

Theorem 1. (Banach's fixed point theorem) *If $f\colon X \to X$ is a contraction mapping on a nonempty complete metric space (X, d), then f has only one fixed point x^*. Moreover, x^* can be found as follows: start with an arbitrary $x_0 \in X$ and define the sequence $x_n = f(x_{n-1})$; then $\lim_{n\to\infty} x_n = x^*$.*

The following well-known statement and its corollary play a crucial role in the proofs.

The derivative of the sigmoid function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{1}{(1 + e^{-\lambda x})},$$

$(\lambda > 0)$ *is bounded by* $\lambda/4$.

Moreover, the mean value theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. This implies that for a log-sigmoid function f we have $|f(x) - f(y)| \leq \lambda/4 \cdot |x - y|$.

In Section 4, sufficient conditions are provided for the existence and uniqueness of fixed points of FGCMs. These conditions are expressed by various norms of a matrix. For convenience, we shortly summarize the matrix norms applied.

Definition 2. The 1-norm (also know as the column norm) of matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$\begin{aligned} \|M\|_1 &= \sup\left\{\frac{\|Mx\|_1}{\|x\|_1}: \quad x \in \mathbb{R}^n,\, x \neq \underline{0}\right\} \\ &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|m_{ij}|, \end{aligned} \tag{7}$$

which is the maximum absolute column sum of the matrix. Here $\|x\|_1$ denotes the 1-norm of vector x, i.e., the sum of the absolute values of its elements.

Definition 3. The ∞-norm (also know as the row norm) of matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$\begin{aligned} \|M\|_\infty &= \sup\left\{\frac{\|Mx\|_\infty}{\|x\|_\infty}: \quad x \in \mathbb{R}^n,\, x \neq \underline{0}\right\} \\ &= \max_{1\leq i\leq n}\sum_{j=1}^{n}|m_{ij}|, \end{aligned} \tag{8}$$

which is the maximum absolute row sum of the matrix. Here $\|x\|_\infty$ denotes the ∞-norm of vector x, i.e., the maximum of the absolute values of its elements.

Definition 4. The Frobenius norm of matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$\|M\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}^2\right)^{1/2}. \tag{9}$$

In our previous work (Harmati *et al.*, 2018) the following theorem was proved regarding the existence and uniqueness of fixed points of FCMs. In Section 4 we will show that for fuzzy grey cognitive maps a very similar result can be derived. Moreover, this theorem is a special case of the results presented in Section 4.

Theorem 2. *Let* W *be the extended (including possible feedback) weight matrix of an FCM, and let* $\lambda > 0$ *be the parameter of the log-sigmoid function. If*

$$\|W\|_F < \frac{4}{\lambda} \tag{10}$$

then the FCM has one and only one fixed point.

We should note that this is a sufficient but not necessary condition. The fact that $\|W\|_F < 4/\lambda$ implies that there is one and only one fixed point, while if $\|W\|_F \geq 4/\lambda$ we do not know whether there are more than one fixed point or limit cycle.

3.2. Fuzzy grey cognitive maps. In many real-life problems, model parameters are based on estimations of human experts; consequently, these estimations carry built-in and unavoidable uncertainties. A possible way of representing these uncertainties is applying intervals instead of crisp numbers (Smoczek, 2013; Vidhya and Hepzibah, 2017). In the field of fuzzy cognitive maps, a similar approach was introduced (Salmeron, 2010) under the name of fuzzy grey cognitive maps.

Fuzzy grey cognitive maps (FGCMs) are effective problem-solving techniques within environments with high uncertainty and imprecision, combining the findings of grey systems theory (GST) and fuzzy cognitive maps (Salmeron, 2010; Papageorgiou and Salmeron, 2012). Further examples for applications of fuzzy grey cognitive maps were introduced by Zanon and Carpinetti (2018). FGCMs were designed to analyze small data samples with poor information (Salmeron and Papageorgiou, 2012; Salmeron and Gutierrez, 2012). The main difference between the fuzzy and grey systems concepts arises in the *intension* and *extension* of the modelled or analyzed object. While grey systems theory focuses on objects with clear extension and unclear intension, in most of the cases fuzzy theory deals with objects with clear intension and unclear extension.

FGCM based models unavoidably consist of numerical operations on the so-called grey numbers. A grey number (usually denoted by $\otimes g$) is the one whose accurate value is unknown, but the range within which the value is included is known. A grey number with both a lower limit ($\underline{g}$) and an upper limit ($\overline{g}$) is called an interval grey number (Liu and Lin, 2006), so $\otimes g \in [\underline{g}, \overline{g}]$. In applications, a grey number is usually an interval. Basic operations on grey numbers are the following:

1. $\otimes g_1 + \otimes g_2 \in [\underline{g_1} + \underline{g_2}, \overline{g_1} + \overline{g_2}]$;
2. $-\otimes g \in [-\overline{g}, -\underline{g}]$;
3. $\otimes g_1 - \otimes g_2 \in [\underline{g_1} - \overline{g_2}, \overline{g_1} - \underline{g_2}]$;
4. $\otimes g_1 \times \otimes g_2 \in [\min(S), \max(S)]$; where $S = \left\{\underline{g_1}\cdot\underline{g_2}, \underline{g_1}\cdot\overline{g_2}, \overline{g_1}\cdot\underline{g_2}, \overline{g_1}\cdot\overline{g_2}\right\}$;
5. if $\alpha > 0$, $\alpha \in \mathbb{R}$, then $\alpha \cdot \otimes g \in [\alpha\underline{g}, \alpha\overline{g}]$.

The following statement is straightforward and plays an important role in our further investigations:

If $f\colon \mathbb{R} \to \mathbb{R}$ *is a monotone increasing function, then* $f(\otimes g) \in [f(\underline{g}), f(\overline{g})]$.

A fuzzy grey cognitive map models causal but uncertain knowledge through grey relationships (grey numbers) between the concepts based on fuzzy cognitive maps. FGCMs are a generalization of FCMs since an FGCM with all the relations' intensities represented by exact numbers (in grey systems theory called white numbers) would be a usual FCM. In general, the FGCM represents human intelligence better than the FCM, because it expresses unclear relations between factors and models incomplete information better than the FCM (Salmeron, 2010).

Example 2. Let us consider the FCM structure given in Example 1, assuming that the weights are not determined precisely, but the intervals containing them are known, i.e., the weight matrix contains grey numbers. One possible scenario is shown below:

$$\otimes W = \begin{bmatrix} 0 & \otimes w_{12} & 0 & 0 & 0 & \otimes w_{16} \\ 0 & 0 & \otimes w_{23} & 0 & 0 & \otimes w_{26} \\ \otimes w_{31} & 0 & 0 & \otimes w_{34} & 0 & 0 \\ \otimes w_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \otimes w_{53} & 0 & 0 & 0 \\ 0 & \otimes w_{62} & 0 & \otimes w_{64} & \otimes w_{65} & 0 \end{bmatrix}, \tag{11}$$

$$\begin{aligned} &\otimes w_{12} \in [0.01, 0.15], && \otimes w_{16} \in [-0.1, -0.5], \\ &\otimes w_{23} \in [-0.8, -0.6], && \otimes w_{26} \in [0.05, 0.15], \\ &\otimes w_{31} \in [0.5, 0.7], && \otimes w_{34} \in [-0.6, -0.2], \\ &\otimes w_{41} \in [0.8, 1], && \otimes w_{53} \in [0.8, 1], \\ &\otimes w_{62} \in [0.1, 0.3], && \otimes w_{64} \in [0.8, 1], \\ &\otimes w_{65} \in [-1, -0.8]. && \end{aligned} \tag{12}$$

♦

The dynamics of an FGCM begin with the initial grey vector state $A(0)$, which represents initial uncertainty. The elements of this vector are grey numbers, i.e., $A_i(0) \in [\underline{A_i(0)}, \overline{A_i(0)}]$ for every i. The updated nodes' states are computed by an iterative process with an activation function, resulting in grey numbers as concept values:

$$A_i(k) \in \left[f(\underline{w_i A(k-1)}), f(\overline{w_i A(k-1)})\right]. \tag{13}$$

Similarly to the classical FCM, after a certain number of iterations, an FGCM with a continuous threshold function arrives at one of the following cases:

1. It settles down to the so-called grey fixed point attractor. This means that the activation vector reaches an equilibrium point; its coordinates become stabilized.
2. The activation vector keeps cycling between several states, which is known as a limit grey cycle.
3. The FGCM continues to produce different grey vector states for each iteration; this is the grey chaotic attractor.

4. Convergence of fuzzy grey cognitive maps

In this section, we provide sufficient conditions for the existence and uniqueness of fixed points of FGCMs. These conditions are expressed by matrix norms and derived applying the contraction mapping theorem with suitable distance metrics. First, the case of a log-sigmoid threshold function is discussed, then the hyperbolic tangent is described and, finally, based on the previous cases, the case of arbitrary sigmoid-like (S-shaped) threshold function is examined.

The updating process (iteration) ensures that a fuzzy cognitive map with grey weights (FGCM) has grey concept values. We assume something about the grey weights; namely, the assumption is that the human expert or the training process assigns the proper signs to the weights, so a weight is either positive or negative (more exactly, nonnegative or nonpositive). This means that the type of relationship (direct or inverse) between the concepts is properly described by the fuzzy cognitive map, so in most cases that seems to be a reasonable assumption.

Let $\otimes w_{ij}$ be a weight describing the connection between concepts C_j and C_i. Due to our assumptions, $\otimes w_{ij}$ is a grey number and it is an element of a subset of the interval $[-1, 0]$ or the interval $[0, 1]$, so

$$\otimes\, w_{ij} \in [\underline{w_{ij}}, \overline{w_{ij}}] \subset [-1, 0] \tag{14}$$

or

$$\otimes\, w_{ij} \in [\underline{w_{ij}}, \overline{w_{ij}}] \subset [0, 1]. \tag{15}$$

Let us introduce the following notation:

$$w^*_{ij} = \begin{cases} |\underline{w_{ij}}| & \text{if } \otimes w_{ij} \le 0, \\ \overline{w_{ij}} & \text{if } \otimes w_{ij} \ge 0, \end{cases} \tag{16}$$

i.e., w^*_{ij} is the absolute value of the most extreme value of the interval containing $\otimes w_{ij}$ possible.

4.1. Log-sigmoid threshold function. In applications of fuzzy cognitive maps, the most widely used threshold function is the log-sigmoid (also known as sigmoid) function, $f(x) = 1/(1 + e^{-\lambda x})$. The following three theorems provide a sufficient condition for the existence and uniqueness of a fixed point of FGCMs, when the threshold function is the log-sigmoid function. Here fixed point $\otimes A^*$ is

$$\begin{aligned}\otimes A^* &= [\otimes A_1^*, \dots, \otimes A_n^*]^T \\ &\in \left[[\underline{A_1^*},\ \overline{A_1^*}], \dots, [\underline{A_n^*},\ \overline{A_n^*}]\right]^T.\end{aligned}$$

The grey fixed point is unique in the sense that the endpoints of the intervals containing grey concept values are unique, i.e., the values $\underline{A_i^*}$ and $\overline{A_i^*}$ are unique for every i.

Theorem 3. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let w_{ij}^* be defined as in Eqn. (16). Moreover, let $\lambda > 0$ be the parameter of the sigmoid function $f(x) = 1/(1+e^{-\lambda x})$ applied to the iteration. If*

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} {w_{ij}^*}^2\right)^{1/2} < \frac{4}{\lambda} \tag{17}$$

then the FGCM has one and only one grey fixed point, regardless of the initial concept values.

Actually, the left handside term is the Frobenius norm of a matrix whose entries are the w_{ij}^* values.

Proof. Let $\otimes A$ be the vector of concept values: $\otimes A = [\otimes A_1, \dots, \otimes A_n]^T$, and let G be a function for the iteration process:

$$\begin{aligned}\otimes G(A) &= G(\otimes A) \\ &= [G(\otimes A)_1, \dots, G(\otimes A)_n]^T \\ &= [\otimes G(A)_1, \dots, \otimes G(A)_n]^T.\end{aligned} \tag{18}$$

Here the elements of the vector are grey numbers, i.e.,

$$\otimes\, G(A)_i \in \left[f(\underline{w_i A}), f(\overline{w_i A})\right], \tag{19}$$

where w_i is the i-th row of matrix $\otimes W$. We are going to show that, for a suitable distance metric d and under certain conditions, the inequality

$$d(\otimes G(A), \otimes G(A')) \le c \cdot d(\otimes A, \otimes A') \tag{20}$$

holds with $0 \le c < 1$, so that mapping G is a contraction, and consequently it has one and only one fixed point. For metric d, we choose the following one:

$$d(\otimes A, \otimes A') = \left[\sum_{i=1}^{n} d^2(\otimes A_i, \otimes A_i')\right]^{1/2}, \tag{21}$$

where

$$d^2(\otimes A_i, \otimes A_i') = \frac{\left(\underline{A_i} - \underline{A_i'}\right)^2 + \left(\overline{A_i} - \overline{A_i'}\right)^2}{2}. \tag{22}$$

Note that, if $\underline{A_i} = \overline{A_i}$ for $\otimes A$ and $\otimes A'$ and for all of the coordinates, then it becomes the ordinary Euclidean metric. Now, we shall give an upper estimate of the distance of $\otimes G(A)$ and $\otimes G(A')$. According to the definition of the distance metric, the square of the distance of $\otimes G(A)$ and $\otimes G(A')$ is the following:

$$\begin{aligned}&d^2(\otimes G(A), \otimes G(A')) \\ &= \sum_{i=1}^{n} d^2(\otimes G(A)_i, \otimes G(A')_i) \\ &= \sum_{i=1}^{n} \frac{\left(\underline{G(A)_i} - \underline{G(A')_i}\right)^2 + \left(\overline{G(A)_i} - \overline{G(A')_i}\right)^2}{2}.\end{aligned} \tag{23}$$

Remember that the lower and upper terms can be expressed by the threshold function (since the threshold function is monotone increasing):

$$\underline{G(A)_i} = \underline{f(w_i A)} = f(\underline{w_i A}), \tag{24}$$

$$\overline{G(A)_i} = \overline{f(w_i A)} = f(\overline{w_i A}). \tag{25}$$

Moreover, the following inequalities hold:

$$\begin{aligned}\left(\underline{G(A)_i} - \underline{G(A')_i}\right)^2 &= \left(f(\underline{w_i A}) - f(\underline{w_i A'})\right)^2 \\ &\le \left(\frac{\lambda}{4}\right)^2 \left(\underline{w_i A} - \underline{w_i A'}\right)^2,\end{aligned} \tag{26}$$

$$\begin{aligned}\left(\overline{G(A)_i} - \overline{G(A')_i}\right)^2 &= \left(f(\overline{w_i A}) - f(\overline{w_i A'})\right)^2 \\ &\le \left(\frac{\lambda}{4}\right)^2 \left(\overline{w_i A} - \overline{w_i A'}\right)^2.\end{aligned} \tag{27}$$

The main problem is that the equations

$$\underline{w_i A} = \sum_{j=1}^{n} \underline{w_{ij}} \cdot \underline{A_j}, \tag{28}$$

$$\overline{w_i A} = \sum_{j=1}^{n} \overline{w_{ij}} \cdot \overline{A_j} \tag{29}$$

hold only in the case when all of weights are nonnegative. In general,

$$\otimes\, w_{ij} A_j \in \begin{cases} \left[\underline{w_{ij}} \cdot \underline{A_j},\ \overline{w_{ij}} \cdot \overline{A_j}\right] & \text{if } \otimes w_{ij} \ge 0, \\ \left[\underline{w_{ij}} \cdot \overline{A_j},\ \overline{w_{ij}} \cdot \underline{A_j}\right] & \text{if } \otimes w_{ij} \le 0. \end{cases} \tag{30}$$

In the case when all of the weights are nonnegative we get the following upper estimate:

$$\begin{aligned}
&\left(f(\underline{w_iA}) - f(\underline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\underline{w_iA} - \underline{w_iA'}\right)^2 \\
&\quad = \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \underline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \underline{w_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\underline{A_j} - \underline{A'_j}\right)^2\right), \qquad (31)
\end{aligned}$$

where the last row comes from applying the well-known Cauchy–Schwarz inequality. A similar inequality is true for the upper endpoints:

$$\begin{aligned}
&\left(f(\overline{w_iA}) - f(\overline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \overline{w_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\overline{A_j} - \overline{A'_j}\right)^2\right). \qquad (32)
\end{aligned}$$

Applying the definition of w^*_{ij}, further upper estimates can be given:

$$\begin{aligned}
&\left(\underline{G(A_i)} - \underline{G(A'_i)}\right)^2 \\
&\quad = \left(f(\underline{w_iA}) - f(\underline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\underline{A_j} - \underline{A'_j}\right)^2\right), \qquad (33)
\end{aligned}$$

$$\begin{aligned}
&\left(\overline{G(A_i)} - \overline{G(A'_i)}\right)^2 \\
&\quad = \left(f(\overline{w_iA}) - f(\overline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\overline{A_j} - \overline{A'_j}\right)^2\right). \qquad (34)
\end{aligned}$$

Now we are ready to give an upper estimate for the distance of $\otimes G(A)$ and $\otimes G(A')$. From Eqn. (23), applying the inequalities (33) and (34), and rearranging and factorizing, we get

$$\begin{aligned}
&d^2(\otimes G(A), \otimes G(A')) \\
&\quad = \sum_{i=1}^{n} d^2(\otimes G(A)_i, \otimes G(A')_i) \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot d^2(\otimes A, \otimes A'). \qquad (35)
\end{aligned}$$

Finally, by taking the square root of each side, we get

$$\begin{aligned}
&d(\otimes G(A), \otimes G(A')) \\
&\qquad \le \frac{\lambda}{4} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2} \cdot d(\otimes A, \otimes A'). \qquad (36)
\end{aligned}$$

By Banach's fixed point theorem, if

$$\frac{\lambda}{4}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2\right)^{1/2} < 1,$$

then mapping G is a contraction, so it has one and only one fixed point, which was the statement in Theorem 3.

Let us turn to the case when all of the weights are nonpositive. The argument is similar to the previous one, with a minor modification:

$$\begin{aligned}
&\left(f(\underline{w_iA}) - f(\underline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\underline{w_iA} - \underline{w_iA'}\right)^2 \\
&\quad = \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \underline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \underline{w_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\overline{A_j} - \overline{A'_j}\right)^2\right). \qquad (37)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(f(\overline{w_iA}) - f(\overline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} \overline{w_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\underline{A_j} - \underline{A'_j}\right)^2\right). \qquad (38)
\end{aligned}$$

Moreover, applying again the definition of w^*_{ij}, we can state that

$$\begin{aligned}
&\left(f(\underline{w_iA}) - f(\underline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\overline{A_j} - \overline{A'_j}\right)^2\right) \qquad (39)
\end{aligned}$$

$$\begin{aligned}
&\left(f(\overline{w_iA}) - f(\overline{w_iA'})\right)^2 \\
&\quad \le \left(\frac{\lambda}{4}\right)^2 \left(\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot \left(\sum_{j=1}^{n} \left(\underline{A_j} - \underline{A'_j}\right)^2\right). \qquad (40)
\end{aligned}$$

From this point the proof goes on the same way as in the previous case and we get the same inequality:

$$\begin{aligned}
&d(\otimes G(A), \otimes G(A')) \\
&\qquad \le \frac{\lambda}{4} \cdot \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2} \cdot d(\otimes A, \otimes A'), \qquad (41)
\end{aligned}$$

In the general case, when positive and negative weights also occur in the weight matrix, in the upper bound of $\left(f(\underline{w_iA}) - f(\underline{w_iA'})\right)^2$ terms like $\underline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)$ and $\underline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)$ may occur. The following facts come trivially from the two cases we have discussed already:

- If $\underline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)$ appears in the upper bound of $\left(f(\underline{w_i A}) - f(\underline{w_i A'})\right)^2$, then $\overline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)$ appears in the upper bound of $\left(f(\overline{w_i A} - f(\overline{w_i A'})\right)^2$.
- If $\underline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)$ appears in the upper bound of the expression $\left(f(\underline{w_i A}) - f(\underline{w_i A'})\right)^2$, then $\overline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)$ appears in the upper bound of $\left(f(\overline{w_i A}) - f(\overline{w_i A'})\right)^2$.

Moreover, the following inequalities come from the definition of w^*_{ij}:

$$\left|\underline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)\right| \le w^*_{ij} \cdot \left|\underline{A_j} - \underline{A'_j}\right|, \tag{42}$$

$$\left|\overline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)\right| \le w^*_{ij} \cdot \left|\overline{A_j} - \overline{A'_j}\right|, \tag{43}$$

$$\left|\underline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)\right| \le w^*_{ij} \cdot \left|\overline{A_j} - \overline{A'_j}\right|, \tag{44}$$

$$\left|\overline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)\right| \le w^*_{ij} \cdot \left|\underline{A_j} - \underline{A'_j}\right|. \tag{45}$$

Applying these upper estimates and then rearranging the terms in the upper estimates of $d^2(\otimes G(A), \otimes G(A'))$ by $\left(\underline{A_j} - \underline{A'_j}\right)^2$ and $\left(\overline{A_j} - \overline{A'_j}\right)^2$, we get

$$\begin{aligned} &d^2(\otimes G(A), \otimes G(A')) \\ &\le \left(\frac{\lambda}{4}\right)^2 \sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2 \cdot \sum_{j=1}^{n} \frac{\left(\underline{A_j} - \underline{A'_j}\right)^2 + \left(\overline{A_j} - \overline{A'_j}\right)^2}{2} \\ &= \left(\frac{\lambda}{4}\right)^2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2\right) \cdot d^2(\otimes A, \otimes A'). \end{aligned} \tag{46}$$

Taking the square root of both the sides, we get the same inequality as in the previous cases:

$$\begin{aligned} &d(\otimes G(A), \otimes G(A')) \\ &\qquad \le \frac{\lambda}{4} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2} \cdot d(\otimes A, \otimes A'). \end{aligned} \tag{47}$$

If

$$\frac{\lambda}{4}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} {w^*_{ij}}^2\right]^{1/2} < 1,$$

then mapping G is a contraction, so according to Banach's fixed point theorem it has one and only one fixed point, which completes the proof for the general case. ■

Note that, if $\underline{w_{ij}} = \overline{w_{ij}}$ for all the weights (i.e., the weights are numbers without uncertainty; in grey systems theory these numbers are not called grey but white numbers), then this condition is the same as in Theorem 2.

Theorem 4. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let $\lambda > 0$ be the parameter of the sigmoid function $f(x) = 1/(1 + e^{-\lambda x})$ applied to the iteration. Let w^*_{ij} be defined as in Eqn. (16). Moreover, let W^* be a matrix defined by the w^*_{ij} values. If*

$$\|W^*\|_1 < \frac{4}{\lambda}, \tag{48}$$

then the FGCM has one and only one grey fixed point, regardless of the initial concept values.

Proof. Just like in the proof of Theorem 3, we are going to show that, for a suitable distance metric d and under certain conditions, the inequality

$$d(\otimes G(A), \otimes G(A')) \le c \cdot d(\otimes A, \otimes A') \tag{49}$$

holds with $0 \le c < 1$, so mapping G is a contraction and it has one and only one fixed point. In this case, for metric d we choose

$$d(\otimes A, \otimes A') = \frac{1}{2}\left(\|\underline{A} - \underline{A'}\|_1 + \left\|\overline{A} - \overline{A'}\right\|_1\right). \tag{50}$$

Here $\|*\|_1$ stands for the 1-norm of the vector, which is the sum of the absolute values of its elements, so in another form the distance metric is

$$d(\otimes A, \otimes A') = \frac{1}{2}\left(\sum_{i=1}^{n}\left|\underline{A_i} - \underline{A'_i}\right| + \sum_{i=1}^{n}\left|\overline{A_i} - \overline{A'_i}\right|\right). \tag{51}$$

Note that if $\underline{A_i} = \overline{A_i}$ for A and A' and for all of the coordinates, then it becomes the so-called Manhattan (or taxicab) metric. The distance of $G(A)$ and $G(A')$ is

$$\begin{aligned} &d(\otimes G(A), \otimes G(A')) \\ &\quad = \frac{1}{2}\Big(\sum_{i=1}^{n}\left|\underline{G(A)_i} - \underline{G(A')_i}\right| \\ &\qquad + \sum_{i=1}^{n}\left|\overline{G(A)_i} - \overline{G(A')_i}\right|\Big). \end{aligned} \tag{52}$$

In the proof of Theorem 3 we saw that

$$\left|\underline{G(A)_i} - \underline{G(A')_i}\right| \le \frac{\lambda}{4}\left|\underline{w_i A} - \underline{w_i A'}\right|, \tag{53}$$

$$\left|\overline{G(A)_i} - \overline{G(A')_i}\right| \le \frac{\lambda}{4}\left|\overline{w_i A} - \overline{w_i A'}\right|. \tag{54}$$

Applying these inequalities, we get an upper estimate for the distance of $\otimes G(A)$ and $\otimes G(A')$:

$$\begin{aligned} &d(\otimes G(A), \otimes G(A')) \\ &\le \frac{\lambda}{8}\left(\sum_{i=1}^{n}\left|\underline{w_i A} - \underline{w_i A'}\right| + \sum_{i=1}^{n}\left|\overline{w_i A} - \overline{w_i A'}\right|\right). \end{aligned} \tag{55}$$

Note that $w_i A = \sum_{j=1}^{n} w_{ij} A_j$ and Eqn. (30) applies here again. In the general case, when positive and negative weights also occur, in the upper bound of $\left|\underline{w_i A} - \underline{w_i A'}\right|$ terms like $\underline{w_{ij}} \cdot \left|\underline{A_j} - \underline{A'_j}\right|$ and $\underline{w_{ij}} \cdot \left|\overline{A_j} - \overline{A'_j}\right|$ may occur. We can observe and justify by simple algebraic rearrangement that

- if $\left|\underline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)\right|$ appears in the upper bound of $\left|\underline{w_i A} - \underline{w_i A'}\right|$, then $\left|\overline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)\right|$ appears in the upper bound of $\left|\overline{w_i A} - \overline{w_i A'}\right|$;
- if $\left|\underline{w_{ij}} \cdot \left(\overline{A_j} - \overline{A'_j}\right)\right|$ appears in the upper bound of $\left|\underline{w_i A} - \underline{w_i A'}\right|$, then $\left|\overline{w_{ij}} \cdot \left(\underline{A_j} - \underline{A'_j}\right)\right|$ appears in the upper bound of $\left|\overline{w_i A} - \overline{w_i A'}\right|$.

Moreover, Eqns. (42)–(45) also hold here. This implies

$$\begin{aligned} & d(\otimes G(A), \otimes G(A')) \\ & \leq \frac{\lambda}{8}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} w^*_{ij} \left|\underline{A_j} - \underline{A'_j}\right| \\ & \quad + \sum_{i=1}^{n}\sum_{j=1}^{n} w^*_{ij} \left|\overline{A_j} - \overline{A'_j}\right|\Big) \\ & = \frac{\lambda}{8}\left(\|W^* \cdot (\underline{A} - \underline{A'})\|_1 + \left\|W^* \cdot (\overline{A} - \overline{A'})\right\|_1\right). \end{aligned} \tag{56}$$

Using the the definition of the 1-norm of a matrix, we get

$$\begin{aligned} \|W^* \cdot (\underline{A} - \underline{A'})\|_1 &= \frac{\|W^* \cdot (\underline{A} - \underline{A'})\|_1}{\|\underline{A} - \underline{A'}\|_1} \|\underline{A} - \underline{A'}\|_1 \\ &\leq \|W^*\|_1 \cdot \|\underline{A} - \underline{A'}\|_1 \end{aligned} \tag{57}$$

The inequality comes from the fact that $\|W^*\|_1$ is defined as the supremum of $\|W^* x\|_1 / \|x\|_1$ (see Definition 2). Applying this upper bound, we get the following inequality for the distance of $\otimes G(A)$ and $\otimes G(A')$:

$$\begin{aligned} & d(\otimes G(A), \otimes G(A')) \\ & \leq \frac{\lambda}{4} \|W^*\|_1 \frac{1}{2}\left(\|\underline{A} - \underline{A'}\|_1 + \left\|\overline{A} - \overline{A'}\right\|_1\right). \end{aligned} \tag{58}$$

Here the last term, $\frac{1}{2}\left(\|\underline{A} - \underline{A'}\|_1 + \left\|\overline{A} - \overline{A'}\right\|_1\right)$, is the distance between A and A', i.e., it is $d(\otimes A, \otimes A')$. According to Banach's fixed point theorem, if the coefficient of $d(\otimes A, \otimes A')$ is less than one, then mapping G is a contraction. It is equivalent to the condition $\|W^*\|_1 < 4/\lambda$ in the theorem, which completes the proof. ■

Theorem 5. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let $\lambda > 0$ be the parameter of the sigmoid function $f(x) = 1/(1 + e^{-\lambda x})$ applied to the iteration. Let w^*_{ij} be defined as in Eqn. (16). Moreover, let W^* be a matrix defined by the w^*_{ij} values. If*

$$\|W^*\|_\infty < \frac{4}{\lambda} \tag{59}$$

then the FGCM has one and only one grey fixed point, regardless of the initial concept values.

Proof. Let the distance of grey concept vectors A and A' be defined as

$$d(\otimes A, \otimes A') = \max_i \max\left\{\left|\underline{A_i} - \underline{A'_i}\right|, \left|\overline{A_i} - \overline{A'_i}\right|\right\}. \tag{60}$$

One can recognize that this is similar to the so-called Hausdorff–Pompeiu distance. In the proof, we apply the contraction mapping theorem using this distance metric:

$$\begin{aligned} & d(\otimes G(A), \otimes G(A')) \\ & = \max_i \max\left\{\left|\underline{G(A)_i} - \underline{G(A')_i}\right|, \left|\overline{G(A)_i} - \overline{G(A')_i}\right|\right\} \\ & \leq \max_i \max\left\{\frac{\lambda}{4}\left|\underline{w_i A} - \underline{w_i A'}\right|, \frac{\lambda}{4}\left|\overline{w_i A} - \overline{w_i A'}\right|\right\}. \end{aligned} \tag{61}$$

For further upper estimation, we use the following inequalities:

$$\sum_{j=1}^{n} w^*_{ij} \left|\underline{A_j} - \underline{A'_j}\right| \leq \sum_{j=1}^{n} w^*_{ij} \max_j \left|\underline{A_j} - \underline{A'_j}\right|, \tag{62}$$

$$\sum_{j=1}^{n} w^*_{ij} \left|\overline{A_j} - \overline{A'_j}\right| \leq \sum_{j=1}^{n} w^*_{ij} \max_j \left|\overline{A_j} - \overline{A'_j}\right|. \tag{63}$$

Thus we get

$$\begin{aligned} & \max\left\{\left|\underline{w_i A} - \underline{w_i A'}\right|, \left|\overline{w_i A} - \overline{w_i A'}\right|\right\} \\ & \leq \sum_{j=1}^{n} w^*_{ij} \max\left\{\left|\underline{A_j} - \underline{A'_j}\right|, \left|\overline{A_j} - \overline{A'_j}\right|\right\}. \end{aligned} \tag{64}$$

Putting these altogether, we have

$$\begin{aligned} & d(\otimes G(A), \otimes G(A')) \\ & \leq \frac{\lambda}{4} \max_i \max\left\{\left|\underline{w_i A} - \underline{w_i A'}\right|, \left|\overline{w_i A} - \overline{w_i A'}\right|\right\} \\ & \leq \frac{\lambda}{4} \max_i \left\{\sum_{j=1}^{n} w^*_{ij} \max\left\{\left|\underline{A_j} - \underline{A'_j}\right|, \left|\overline{A_j} - \overline{A'_j}\right|\right\}\right\} \\ & \leq \frac{\lambda}{4} \max_i \left\{\sum_{j=1}^{n} w^*_{ij}\right\} \\ & \quad \cdot \max_j \max\left\{\left|\underline{A_j} - \underline{A'_j}\right|, \left|\overline{A_j} - \overline{A'_j}\right|\right\}. \end{aligned} \tag{65}$$

From the fact that $w^*_{ij} \geq 0$ and from the definition of the distance applied, we get

$$\|W^*\|_\infty = \max_i \left\{ \sum_{j=1}^{n} w^*_{ij} \right\}, \tag{66}$$

$$d(\otimes A, \otimes A') = \max_j \max \left\{ \left| \underline{A_j} - \underline{A'_j} \right|, \left| \overline{A_j} - \overline{A'_j} \right| \right\}. \tag{67}$$

Finally, we obtain

$$d(\otimes G(A), \otimes G(A')) \leq \frac{\lambda}{4} \|W^*\|_\infty d(\otimes A, \otimes A'). \tag{68}$$

If the coefficient of the distance between A and A' is less than one, then this mapping is a contraction, and consequently it has exactly one fixed point. This is equivalent to the statement that $\|W^*\|_\infty < 4/\lambda$, which was the condition in the theorem. ■

4.2. Hyperbolic tangent threshold function. In some special cases, the activation values may have negative and positive values, too, so the possible values come from the interval $[-1, 1]$ (instead of the interval $[0, 1]$). Since the range of the log-sigmoid function is the interval $(0, 1)$, it is not applicable here as a threshold function. The most widely used function in this situation is the hyperbolic tangent, which has the range $(-1, 1)$:

$$\tanh(\lambda x) = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1}. \tag{69}$$

Here parameter λ is responsible for the steepness of the function: the larger the value of λ, the steeper the function.

Theorem 6. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let w^*_{ij} be defined as in Eqn. (16). Moreover, let $\lambda > 0$ be the parameter of the hyperbolic tangent function*

$$f(x) = \tanh(\lambda x) = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1}$$

applied for the iteration. If

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{n} {w^*_{ij}}^2 \right)^{1/2} < \frac{1}{\lambda} \tag{70}$$

then the FGCM has one and only one fixed point, regardless of the initial concept values.

Proof. The proof is similar to the case of the log-sigmoid function (i.e., we prove that the mapping is a contraction with a suitable distance metric), but there are some differences:

1. The distance metric applied is the same as in the proof of Theorem 3.

2. The maximal value of the derivative of the threshold function $\tanh(\lambda x)$ ($\lambda > 0$) is λ, and consequently $|f(x) - f(y)| \leq \lambda \cdot |x - y|$. This inequality introduces the term $1/\lambda$ instead of $4/\lambda$.

3. Based on the previous statement, we have

$$\left(\underline{G(A)_i} - \underline{G(A')_i} \right)^2 \leq \lambda^2 \left(\underline{w_i A} - \underline{w_i A'} \right)^2, \tag{71}$$

$$\left(\overline{G(A)_i} - \overline{G(A')_i} \right)^2 \leq \lambda^2 \left(\overline{w_i A} - \overline{w_i A'} \right)^2. \tag{72}$$

4. Since now the range of the activation values $A_1, \dots, A_n$ is the interval $[-1, 1]$, the upper estimations of terms $|\underline{w_i A} - \underline{w_i A'}|$ and $|\overline{w_i A} - \overline{w_i A'}|$ become more tedious.

   The coordinate $\otimes w_i A \in [\underline{w_i A}, \overline{w_i A}]$ is the sum of grey numbers $\otimes w_{ij} A_j$, $j = 1, \dots, n$, and for the left and right endpoints of the containing intervals we have $\underline{w_i A} = \sum_{i=1}^n \underline{w_{ij} A_j}$ and $\overline{w_i A} = \sum_{i=1}^n \overline{w_{ij} A_j}$. The grey number $\otimes w_{ij} A_j$ can be represented as an element of one of the following intervals:

   if $\otimes w_{ij} \geq 0$,

$$\begin{aligned} &\left[\overline{w_{ij}} \cdot \underline{A_j}, \ \overline{w_{ij}} \cdot \overline{A_j} \right] && \text{if } \underline{A_j} < 0 \text{ and } \overline{A_j} > 0, \\ &\left[\underline{w_{ij}} \cdot \underline{A_j}, \ \overline{w_{ij}} \cdot \overline{A_j} \right] && \text{if } \otimes A_j > 0, \\ &\left[\overline{w_{ij}} \cdot \underline{A_j}, \ \underline{w_{ij}} \cdot \overline{A_j} \right] && \text{if } \otimes A_j < 0; \end{aligned} \tag{73}$$

   if $\otimes w_{ij} \leq 0$,

$$\begin{aligned} &\left[\underline{w_{ij}} \cdot \overline{A_j}, \ \underline{w_{ij}} \cdot \underline{A_j} \right] && \text{if } \underline{A_j} < 0 \text{ and } \overline{A_j} > 0, \\ &\left[\underline{w_{ij}} \cdot \overline{A_j}, \ \overline{w_{ij}} \cdot \underline{A_j} \right] && \text{if } \otimes A_j > 0, \\ &\left[\overline{w_{ij}} \cdot \overline{A_j}, \ \underline{w_{ij}} \cdot \underline{A_j} \right] && \text{if } \otimes A_j < 0. \end{aligned} \tag{74}$$

5. Simple algebraic transformations show that

   if $\otimes w_{ij} \geq 0$,

$$|\underline{w_{ij} A_j} - \underline{w_{ij} A'_j}| \leq w^*_{ij} |\underline{A_j} - \underline{A'_j}|, \tag{75}$$

$$|\overline{w_{ij} A_j} - \overline{w_{ij} A'_j}| \leq w^*_{ij} |\overline{A_j} - \overline{A'_j}|; \tag{76}$$

   if $\otimes w_{ij} \leq 0$,

$$|\underline{w_{ij} A_j} - \underline{w_{ij} A'_j}| \leq w^*_{ij} |\overline{A_j} - \overline{A'_j}|, \tag{77}$$

$$|\overline{w_{ij} A_j} - \overline{w_{ij} A'_j}| \leq w^*_{ij} |\underline{A_j} - \underline{A'_j}|. \tag{78}$$

Here the left and right endpoints $\underline{w_{ij}A_j}$ and $\overline{w_{ij}A_j}$ are from the possibilities described in Eqns. (73) and (74).

6. We can recognize that

- if $w_{ij}^*|\underline{A_j} - \underline{A'_j}|$ appears in the upper estimate of $|\underline{w_{ij}A_j} - \underline{w_{ij}A'_j}|$, then $w_{ij}^*|\overline{A_j} - \overline{A'_j}|$ appears in the upper estimate of $|\overline{w_{ij}A_j} - \overline{w_{ij}A'_j}|$;
- if $w_{ij}^*|\overline{A_j} - \overline{A'_j}|$ appears in the upper estimate of $|\underline{w_{ij}A_j} - \underline{w_{ij}A'_j}|$, then $w_{ij}^*|\underline{A_j} - \underline{A'_j}|$ appears in the upper estimate of $|\overline{w_{ij}A_j} - \overline{w_{ij}A'_j}|$.

This implies that (similarly to the proof of Theorem 3) we can rearrange the terms in the upper estimates of $d^2(\otimes G(A), \otimes G(A'))$ by $(\underline{A_j} - \underline{A'_j})^2$ and $(\overline{A_j} - \overline{A'_j})^2$.

Putting these together, we get

$$\begin{aligned}
&d^2(\otimes G(A), \otimes G(A')) \\
&= \sum_{i=1}^{n} \frac{\left(\underline{G(A)_i} - \underline{G(A')_i}\right)^2 + \left(\overline{G(A)_i} - \overline{G(A')_i}\right)^2}{2} \\
&\le \sum_{i=1}^{n} \frac{\lambda^2 \left(\underline{w_iA} - \underline{w_iA'}\right)^2 + \lambda^2 \left(\overline{w_iA} - \overline{w_iA'}\right)^2}{2} \\
&\le \lambda^2 \sum_{i=1}^{n} \sum_{j=1}^{n} {w_{ij}^*}^2 \frac{(\underline{A_j} - \underline{A'_j})^2 + (\overline{A_j} - \overline{A'_j})^2}{2} \\
&= \lambda^2 \sum_{i=1}^{n} \sum_{j=1}^{n} {w_{ij}^*}^2 \sum_{j=1}^{n} \frac{(\underline{A_j} - \underline{A'_j})^2 + (\overline{A_j} - \overline{A'_j})^2}{2} \\
&= \lambda^2 \sum_{i=1}^{n} \sum_{j=1}^{n} {w_{ij}^*}^2 d^2(\otimes A, \otimes A'). \qquad (79)
\end{aligned}$$

By taking the square root of both the sides, we get

$$d(\otimes G(A), \otimes G(A')) \le \lambda \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} {w_{ij}^*}^2} \cdot d(\otimes A, \otimes A'). \qquad (80)$$

If $\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} {w_{ij}^*}^2} < 1/\lambda$, then the coefficient of $d(\otimes A, \otimes A')$ is less then one, which implies that mapping G is contraction, so it has exactly one fixed point. ■

Theorem 7. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let w_{ij}^* be defined as in Eqn. (16). Let $\lambda > 0$ be the parameter of the hyperbolic tangent function $f(x) = \tanh(\lambda x)$ applied for the iteration. Moreover, let W^* be a matrix defined by the w_{ij}^* values. If*

$$\|W^*\|_1 < \frac{1}{\lambda} \qquad (81)$$

then the FGCM has one and only one fixed point, regardless of the initial concept values.

Proof. The proof follows from those of Theorems 4 and 6. ■

Theorem 8. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let w_{ij}^* be defined as in Eqn. (16). Let $\lambda > 0$ be the parameter of the hyperbolic tangent function $f(x) = \tanh(\lambda x)$ applied to the iteration. Moreover, let W^* be a matrix defined by the w_{ij}^* values. If*

$$\|W^*\|_\infty < \frac{1}{\lambda} \qquad (82)$$

then the FGCM has one and only one fixed point, regardless of the initial concept values.

Proof. The proof follows from those of Theorems 5 and 6. ■

Remark 1. A fuzzy cognitive map equipped with a hyperbolic tangent threshold function always has a fixed point, regardless of the parameter λ and the matrix W. If $A = [0, \dots, 0]^T$, then for every $i \in \{1, \dots, n\}$ we have

$$f(w_iA) = \tanh(\lambda \cdot w_iA) = \tanh(0) = 0, \qquad (83)$$

so $A = [0, \dots, 0]^T$ is always a fixed point. However, it is not always a fixed point *attractor*, i.e., it is not always a stable fixed point.

Since the equilibrium point (if it exists) of a fuzzy cognitive map is determined by iteration, this point is necessarily an attractor (but not necessarily unique; it may depend on the initial guess of activation values). The previous theorems provide sufficient conditions for the existence and uniqueness of fixed points. The existence is clear, as we have seen. The theorems tell us that, if the conditions hold, then the FCM has exactly one fixed point attractor (and we know that it is the crisp point $A = [0, \dots, 0]^T$), regardless of the parameter λ and weight matrix W.

The information provided for the decision-maker by the theorems is that, under the conditions stated, the FGCM will always converge to the zero equilibrium point.

4.3. Arbitrary sigmoid-like threshold function.

Although the most widely used continuous threshold functions in FCM theory and applications are the log-sigmoid and hyperbolic tangent functions, there are other possibilities, too.

From a mathematical point of view, in the general sense, a sigmoid function (S-shaped function) is a bounded, monotone increasing and continuously differentiable real function that is defined for all real values. Some examples (besides the already discussed log-sigmoid and hyperbolic tangent) are the arctangent, error function, cumulative distribution function (cdf) of the normal distribution, Gompertz function, etc.

Let f be a sigmoid, continuously differentiable function, and let K be the maximum value of its derivative. This maximum exists, since f is bounded and monotone increasing. From the mean-value theorem it follows that, for every x and y, $|f(x) - f(y)| \leq K \cdot |x-y|$. If this sigmoid function is applied as an activation (threshold) function for a fuzzy grey cognitive map, then the following theorem provides sufficient conditions for the existence and uniqueness of the fixed point of this FGCM.

Theorem 9. *Let $\otimes W$ be the extended (including possible feedback) weight matrix of a fuzzy grey cognitive map (FGCM), where the weights $\otimes w_{ij}$ are nonnegative or nonpositive grey numbers, and let $f(x)$ be a sigmoid function applied for the iteration, let K be the maximal value of $f'(x)$. Let w^*_{ij} be defined as in Eqn. (16). Moreover, and W^* be a matrix defined by the w^*_{ij} values. If at least one the inequalities*

$$\|W^*\|_F < \frac{1}{K}, \tag{84}$$

$$\|W^*\|_1 < \frac{1}{K}, \tag{85}$$

$$\|W^*\|_\infty < \frac{1}{K} \tag{86}$$

holds, then the fuzzy grey cognitive map has one and only one grey fixed point, regardless of the initial concept values. This grey fixed point is unique in the sense that the endpoints of the intervals containing grey concept values are unique, i.e., the values $\underline{A_i^*}$ and $\overline{A_i^*}$ are unique for every i.

Proof. The proof is a direct adaptation of those of Theorems 3–8. ■

Example 3. Consider the grey weight matrix, introduced in Example 2. The matrix W^*, formed by the w^*_{ij} values (cf. Eqn. (16)), will be the following:

$$W^* = \begin{bmatrix} 0 & 0.15 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0 & 0 & 0.15 \\ 0.7 & 0 & 0 & 0.6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 1 & 1 & 0 \end{bmatrix}. \tag{87}$$

The matrix norms applied in the theorems are

$$\|W^*\|_F = 2.4238 \qquad \|W^*\|_1 = 1.8 \qquad \|W^*\|_\infty = 2.3. \tag{88}$$

This implies, for example, for the case of the log-sigmoid threshold function, that if $\lambda < 2.2222$ then the FGCM has exactly one grey fixed point in the sense that the endpoints of the intervals containing grey concept values are unique. ♦

5. Summary

Fuzzy grey cognitive maps are extensions of fuzzy cognitive maps that are able to handle uncertainties in the weight matrix. These uncertainties can express the imprecision or incomplete information of human experts.

Similarly as in standard FCMs, the inference is based on the long-term behaviour of an iteration process. FGCMs may converge to an equilibrium point (fixed point), produce limit cycles or show a chaotic pattern. In this paper, we provided several sufficient conditions for the convergence of FGCMs to a unique fixed point, regardless of the initial concept values. The fixed point is unique in the sense that the endpoints of the intervals containing the so-called grey concept values are unique. The cases of the log-sigmoid and the hyperbolic tangent threshold functions were discussed in detail. Moreover, the generalizations of the results to arbitrary sigmoidal (S-shaped) threshold functions were also given. The convergence conditions are expressed by the elements of the weight matrix and the maximal value of the derivative of the threshold function, which can be easily expressed by its parameter(s) in most cases. Consequently, these sufficient conditions can be easily checked before simulations, providing assistance for the design of FGCM-based decision support tools.

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