Int. J. Appl. Math. Comput. Sci., 2021, Vol. 31, No. 1, 155–164
DOI: 10.34768/amcs-2021-0011

A COMMUNICATION NETWORK ROUTING PROBLEM: MODELING AND OPTIMIZATION USING NON–COOPERATIVE GAME THEORY

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We consider a communication network routing problem wherein a number of users need to efficiently transmit their throughput demand in the form of data packets (incurring less cost and less delay) through one or more links. Using the game theoretic perspective, we propose a dynamic model which ensures unhindered transmission of data even in the case where the capacity of the link is exceeded. The model incorporates a mechanism in which users are appropriately punished (with additional cost) when the total data to be transmitted exceeds the capacity of the link. The model has multiple Nash equilibrium points. To arrive at rational strategies, we introduce the concept of focal points and get what is termed focal Nash equilibrium (FNE) points for the model. We further introduce the concept of preferred focal Nash equilibrium (PFNE) points and find their relation with the Pareto optimal solution for the model.

Keywords: communication network, routing problem, game theory, focal points, Nash equilibrium, Pareto optimal solution.

1. Introduction

The dynamics of modern networks are well understood by game theoretical models to deal with the routing problem in networking (Orda *et al.*, 1993). Here the term "dynamics" refers to a situation where users change their behavior based on the state of the network. A model for a two-node parallel link communication system was developed for multiple competing users (Sahin and Simaan, 2006). The authors derived flow and routing control policies for each user to get a Nash equilibrium point(s). Conditions for the uniqueness of the Nash equilibrium are also established (Altman *et al.*, 2002). A number of areas are highlighted in which common features between transportation and telecommunication network model existed (Altman and Wynter, 2004).

Routing games were also studied as bottleneck routing games (Banner and Orda, 2007). The authors investigated the fact that "bottleneck" (worst) routing games appear in two main routing scenarios, namely, when a user can split its traffic over more than one path (splitable bottleneck game) and when it (we use the neutral gender for users) cannot (unsplitable bottleneck game). They showed that a bottleneck game always admits a Nash equilibrium; moreover, best response dynamics in unsplitable games converge to a Nash equilibrium in finite time. This Nash equilibrium (both in splitable and unsplitable bottleneck games) can, however, be very inefficient. In order to cope with this inefficiency, the authors investigated, for each game, "reasonable" conditions under which Nash equilibria were socially optimal, i.e., when all users routed their traffic along paths with a minimum number of bottlenecks.

A telecommunication model was also described using queuing theory (Massey, 2002). In this model the impact of time varying behavior on the communication system was studied. Many researchers have used

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game theory to model and resolve the problems of communication networks. An agent based study of a network was also modeled using game theory. Here, the problem of the network and product agents who have conflicting interests was resolved and integrated using cooperative game theory (Nguyen *et al.*, 2013). The game theoretic view of the network provides an equilibrium state of the network having rational users. Attackers change the equilibrium characteristics and hence this attack could be detected. Thus, using this concept, game theory deals with the security problem in networks (Ignatenko, 2016). A probabilistic routing scheme based on game theory for opportunistic networks was also developed (Qin *et al.*, 2019). In this model the messages are forwarded by participating nodes that will then be rewarded. Virtual money is used for this mechanism. This process of forwarding messages was modeled as a bargain game.

The model of transmission in the form of a continuous flow of data over a multiple-link network has also been described (Orda *et al.*, 1993). The links have fixed capacity. The situation where the flow desired by the users together exceeds the capacity of the link is ruled out by making the cost infinite to each of the users. In such a situation, in effect, none of the users is able to transmit its flow irrespective of the size of an individual's desired transmission. Thus, even if a user is wishing to transmit a small amount of data flow, it may not be able to do so because of the greed of other users (resulting in exceeding the link capacity).

In the presented model, an effort has been made to fill in this gap. In its simplistic form, we consider a transmission network consisting of a single link wherein each user is allowed to transmit its flow (in the form of data packets) in a fixed number of discrete time slots. For each time slot, the capacity of the link is considered to be fixed. From the practical viewpoint, where late transmission of data packets may result in a loss and hence an additional cost to the user, we let the transmission cost increase with time. Of course, as usual, the cost function will also be increasing as a function of individual's desired flow size (number of data packets) at a given time slot. Each user is assumed to be aware of this. From the game theoretic point of view, then, this leads to a competitive game. Each user will now have a strategy to transmit as many data packets as early (time slots) as possible. Thus each user is expected to plan a strategy of transmitting its entire flow of data packets over the given number of time slots, keeping in mind the capacity of the link, the total number of slots, and the cost of transmission. The strategy of each user will now be an m-tuple with integer entries as the number of data packets the user intends to transmit in that time slot, where m is the total number of slots.

In the absence of any communication or cooperation, the users will not be aware of each other's strategies. A situation may soon develop where the total flow (total of the data packets) to be transmitted by all the users together at a given slot exceeds the capacity. The mechanism that we propose is that, even in such a situation, the net flow equaling the capacity of the link is transmitted with different users being able to transmit different amounts of data packets depending on their intended flow size. The entire data (packets) of the user desiring to transmit the least amount of data in that slot is transmitted, while the data of the users planning to transmit a higher amount of data will be transmitted only partially in some proportion of their proposed flow (explained in the mathematical model below). The leftover from such a slot for such a user is accommodated/added to the amount of data that the user has planned to transmit in the next time slot. This rule is then applied for each subsequent time slot. This mechanism, however, leads to a proportional increase in the cost to different users. The model thus punishes greedy users proportionally without failing transmission of data in any slot. The actual data transmission that will take place over different time slots for a user may thus not be the same as that proposed by the user in the form of its initial strategy set.

2. Mathematical model

To explain the proposed model mathematically, we consider two users sharing one link connecting a source node to a destination node. We assume that the link is available to the users over a discrete time span known as time slots and there are m time slots in a single cycle with a fixed λ as the capacity of the link for each time slot. We allow the cost function to increase in both the size of data transmission and time (slot). With users expected to be rational/ selfish with the knowledge of cost function, the situation resembles a competitive game. Each user, $n(=1,2)$, has throughput demand $D^{(n)}$ which it wants to transmit from source to destination distributed over m slots in a single cycle. Thus each user comes up with an m-tuple representing its initial strategy.

2.1. Mechanism for data packet transmission. Let $\{p_1, p_2, p_3, \ldots, p_m\}$ and $\{q_1, q_2, q_3, \ldots, q_m\}$ be the initial strategy sets for the two users, respectively. That is, the first user plans to transmit p_i packets while the second user plans to transmit q_i packets in the i-th slot, $1 \le i \le m$. Obviously, $p_i \le \lambda, q_i \le \lambda, 1 \le i \le m$.

If $p_1 + q_1 \le \lambda$, then packets p_1 of user 1 and q_1 of user 2 will be transmitted in the first slot with appropriate cost to each of them. Therefore, we now consider the case $p_1 + q_1 > \lambda$. An appropriate number of packets of each user would be transmitted depending on the relation between p_1 and q_1. The rule to be followed is that all the data packets of the user with minimum packets intended to be transmitted will be transmitted while only those many packets of the other user will be transmitted equaling the remaining capacity of the link. For example, if $p_1 > q_1$, then q_1 packets of User 2, will be transmitted, while only $\lambda - q_1$ packets of User 1 will be transmitted from its intended p_1 packets. The remaining $p_1 - (\lambda - q_1)$ packets will have to be accommodated in the next time slot. Therefore, while q_2 will remain unchanged, p_2 will now increase to $p_2 + (p_1 - (\lambda - q_1))$. If $p_1 = q_1$ (and, of course, $p_1 + q_1 > \lambda$), then $\lambda/2$ packets of each user will be transmitted while both p_2 and q_2 will increase to $p_2 + (p_1 - (\lambda/2))$ and $q_2 + (q_1 - (\lambda/2))$, respectively. Same logic will be followed at each subsequent time slot. Thus, according to this mechanism, not only may the values of p_i and q_i keep on changing while the cycle is in progress, but the actual data packets that may get transmitted at the i-th slot may also not indeed be equal to p_i and q_i.

To explain this succinctly, we need to introduce new sets in addition to p_i's and q_i's. Thus, we shall denote by $\{r_i\}_{i=1}^m$ and $\{s_i\}_{i=1}^m$ the intermediate/modified values of $\{p_i\}$ and $\{q_i\}$, respectively. We shall let $\{u_1, u_2, u_3, \ldots, u_m\}$ and $\{v_1, v_2, v_3, \ldots, v_m\}$ denote the data packets actually transmitted for Users 1 and 2, respectively, over the entire cycle. Obviously, we shall always have $r_1 = p_1$ and $s_1 = q_1$.

It should be clear that while the values of p_i and q_i are initially specified by the user itself, the values of r_i, u_i and s_i, v_i get generated as the cycle progresses over the m time slots.

With the help of this new notation, the cases discussed above can be expressed mathematically as follows:

Case I. When $p_1 + q_1 \leq \lambda$, we have $r_1 = u_1 = p_1$; $s_1 = v_1 = q_1$.

Case II. When $p_1 + q_1 > \lambda$:

(a) if $p_1 > q_1$, then

$$s_1 = v_1 = q_1, \quad u_1 = \lambda - q_1 = \lambda - s_1,$$
$$s_2 = q_2, \quad r_2 = p_2 + (r_1 - u_1);$$

(b) if $q_1 > p_1$, then

$$r_1 = u_1 = p_1, \quad v_1 = \lambda - p_1 = \lambda - r_1,$$
$$r_2 = p_2, \quad s_2 = q_2 + (s_1 - v_1);$$

(c) if $p_1 = q_1$, then

$$u_1 = v_1 = \frac{\lambda}{2},$$
$$r_2 = p_2 + \Big(p_1 - \frac{\lambda}{2}\Big), \quad s_2 = q_2 + \Big(q_1 - \frac{\lambda}{2}\Big).$$

The same logic will be applied to r_i and s_i stepwise for $2 \leq i \leq m$. In general, as r_i and s_i may be greater than p_i and q_i, respectively, the actual cost incurred by a user may be greater than the expected cost corresponding to its initial planned strategy (given by $\{p_i\}$ or $\{q_i\}$).

2.2. Strategy sets and constraints. Some of the simple properties of p_i, r_i, u_i and q_i, s_i, v_i following from the mathematics described above are as follows:

P_1 : $p_i, q_i, r_i, s_i, u_i, v_i \geq 0, \forall\ i = 1, 2, \ldots, m$ (non-negativity constraint),

P_2 : $\sum_i p_i = D^{(1)}$ and $\sum_i q_i = D^{(2)}$ (demand constraints for the first and the second user),

P_3 : $p_i \leq r_i$ and $q_i \leq s_i$,

P_4 : $r_i = p_i + (r_{i-1} - u_{i-1})$ and $s_i = q_i + (s_{i-1} - v_{i-1})$, $m \geq i \geq 2$ (recurrence relations of r_i and s_i),

P_5 : $\lambda_i = u_i + v_i \leq \lambda$, $\forall\ i = 1, 2, \ldots, m$ (capacity constraint for each i),

P_6 : $\sum_i u_i \leq D^{(1)}$ and $\sum_i v_i \leq D^{(2)}$.

It may happen at the end of the cycle, that is, after the final m-th time slot, that a few data packets of a user may be left which could not be transmitted in the m-th time slot, i.e., when $r_m + s_m$ exceeds λ. Since the values of $\{r_i\}$ and $\{s_i\}$ are determined by $\{p_i\}$ and $\{q_i\}$, we can rule out such strategy sets by assigning a very high total cost of data packet transmission after the m-th slot. Thus, from the point of view of cost minimization, such strategy sets would be irrelevant or non-feasible. The last constraint P_6 may thus be replaced by

P_6'**:** $\sum_i u_i = D^{(1)}$ and $\sum_i v_i = D^{(2)}$,

i.e., both the users are able to transmit all their data packets. Since over the complete cycle at most $m\lambda$ data packets can be transmitted, the case where $D^{(1)} + D^{(2)} < m\lambda$ will mean the underutilized capacity of the link. Thus, to make the game more realistic and competitive, we shall consider only the case $D^{(1)} + D^{(2)} = m\lambda$.

3. Cost function and the Nash equilibrium

We shall now discuss the kind of cost functions that are suitable for the model, and see if the very basic conditions on such a cost function can lead to results about the existence of Nash equilibrium strategies.

3.1. General assumptions on the cost function. We shall denote by $C^{(n)}$ the total cost accrued to the n-th user over an entire cycle consisting of m number of time slots, i.e.,

$$C^{(n)} = \sum_{i=1}^{m} C_i^{(n)}.$$

On the lines of the earlier work presented by Orda *et al.* (1993), the following general assumptions on the cost function $C_i^{(n)}$ will be imposed. Here, $C_i^{(n)}$ is the cost borne by the n-th user over the i-th time slot.

A_1: Cost function $C_i^{(n)}$ is a non-negative function of three variables i, x_i and λ_i, where $i(=$ time slot$), x_i(= u_i$ or v_i depending on $n = 1$ or 2) and $\lambda_i(= u_i + v_i)$.

A_2: $C_i^{(n)}$ is strictly increasing with respect to all the three arguments. Although it is not of any particular significance in the subsequent discussion in this paper, we may explicitly separate the dependence on the time slot from the dependence on the number of packets transmitted to write

$$C_i^{(n)} = f(i) \cdot \phi(x_i, \lambda_i), \tag{1}$$

where $i = 1, 2, \ldots, m$.

A_3: $C_i^{(n)}$ is sufficiently smooth.

A_4: $C_i^{(n)}$ is zero when x_i is zero, i.e., if no data packets of a user are transmitted in a time slot, then that particular user does not bear any cost for that slot. This condition does not significantly affect the arguments presented here. Indeed, a user may be made to bear a certain minimum cost for every time slot and this can be achieved by slightly modifying the cost function and without affecting any of the above properties.

3.2. Nash equilibrium. It should be noted from the definition of the cost function that cost to a user for any time slot i depends on the numbers u_i and v_i of the actual packets transmitted by the users in that slot. Since u's and v's are derivable from p's and q's forming the strategy sets of the two users, we can still treat the costs incurred to be functions of the initial strategy sets.

By the definition, the combination of strategies $\{\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \ldots, \tilde{p_m}\}$ for User 1 and $\{\tilde{q_1}, \tilde{q_2}, \tilde{q_3}, \ldots, \tilde{q_m}\}$ for User 2 correspond to the Nash equilibrium point provided

$$\begin{aligned} &C^{(1)}(\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \ldots, \tilde{p_m}, \tilde{q_1}, \tilde{q_2}, \tilde{q_3}, \ldots, \tilde{q_m}) \\ &\quad \le C^{(1)}(p_1, p_2, p_3, \ldots, p_m, \tilde{q_1}, \tilde{q_2}, \tilde{q_3}, \ldots, \tilde{q_m}) \end{aligned} \tag{2}$$

for all possible strategies $\{p_1, p_2, p_3, \ldots, p_m\}$ of User 1 and

$$\begin{aligned} &C^{(2)}(\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \ldots, \tilde{p_m}, \tilde{q_1}, \tilde{q_2}, \tilde{q_3}, \ldots, \tilde{q_m}) \\ &\quad \le C^{(2)}(\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \ldots, \tilde{p_m}, q_1, q_2, q_3, \ldots, q_m) \end{aligned} \tag{3}$$

for all possible strategies $\{q_1, q_2, q_3, \ldots, q_m\}$ of User 2.

If $p_i + q_i = \lambda$, $\forall i$, then all data packets will be transmitted without any penalty.

Mathematically this situation can be expressed as

$$p_i = r_i = u_i, \quad q_i = s_i = v_i, \quad 1 \le i \le m. \tag{4}$$

Each (and only) of these strategy sets will therefore be Nash equilibrium strategies.

The model described in this paper thus represents a game with multiple Nash equilibrium points.

4. Focal points and Pareto optimality

Since the two users in this network model may not be expected to be communicating or cooperating, the question then is the following: What strategy sets may be preferred by the users as rational human beings? To answer this question, we consider the concept of "focalness."

The theory of "focal points" was introduced by Schelling (1960). This theory suggests that in some "real life" situations players may be able to coordinate on a particular equilibrium by using information that is abstracted away by the strategic form. The "focalness" of various strategies depends on the players'/users' past experiences, social norms and information provided to them.

For example, if two cars are being driven in opposite directions on a road, and if head on collision is to be avoided, then (R, R) or (L, L), i.e., both driving on their right or both driving on their left would be equilibrium strategies. However, if they are driving in India, then, because of the traffic rules of driving on one's left, the (L, L) strategy would be preferred over the (R, R) one. This has happened because of their traffic rule knowledge. Thus, although (R, R) and (L, L) are two Nash equilibrium points, (L, L) is the focal point strategy. Similarly, consider a game consisting of two players, each of whom is shown a frame of four colored rectangles and is asked to choose one rectangle. They are rewarded if both select the rectangle of the same color. Suppose each one is shown a frame consisting of three blue rectangles and one red rectangle. Obviously, (b, b) or (r, r) would be benefitting strategies, with greater probability of (b, b) being chosen. However, if the culture of the players has a special relevance for, say, the red color, then instead of (b, b) they will end up with (r, r) choice. The choice has thus been governed by the cultural/social background, and (r, r) then represents a focal point strategy. Unlike in both the examples given above, in a general scenario, a focal point strategy need not necessarily be a Nash equilibrium one. This will become clear in the subsequent discussion.

Now we will try to find out answers to the following queries based on our model:

Q1. What concept of focalness is relevant to the situation depicted by the model and what are the corresponding focal point strategies?

Q2. In the case of a plenty of focal point strategies, can some of them also be Nash equilibrium strategies?

In the present network problem, one can use the rational/selfish thinking of the users as the background that will lead to their focal point strategies. Since the cost of transmission increases with the time and number of data packets to be transmitted, as a motive to minimize individual cost a user may be tempted to have an m-tuple strategy, which is heavily loaded (large values) with the initial entries. If both the players follow this strategy, then from the initial time slots themselves the capacity of the link will get exceeded, resulting in penalty and thus additional cost to both. These kinds of strategies may thus be ruled out as being prone to be costly.

Another possible way of distributing the throughput demand over the m time slots is to use the extended pigeonhole principle.

The Extended pigeonhole principle states that "If n pigeons are assigned to m pigeonholes then one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons."

Although the pigeonhole principle does not rule out zero pigeons or less than average pigeons in one or more holes, in the present scenario, however, this strategy may lead to additional cost to an individual user. Thus, in all likelihood, a user may be tempted to distribute its throughput demand more or less uniformly. Let

$$\beta_1 = \left\lfloor \frac{D^{(1)}-1}{m} \right\rfloor, \quad \beta_2 = \left\lfloor \frac{D^{(2)}-1}{m} \right\rfloor,$$

$$\alpha_1 = D^{(1)} - \beta_1 m, \quad \alpha_2 = D^{(2)} - \beta_2 m. \tag{5}$$

Since $D^{(1)} + D^{(2)} = m\lambda$, it follows that

$$\alpha_1 + \alpha_2 = (\lambda - \beta_1 - \beta_2)m. \tag{6}$$

User 1 may thus choose to distribute its throughput demand $D^{(1)}$ over the m slots as β_1 data packets in each of the m slots and remaining α_1 data packets distributed as additional over the m slots as per its wish. User 2 may also choose to have a similar distribution of its throughput demand.

We shall call a strategy of User 1 to be a focal point strategy if the strategy is given by an m-tuple having entry values greater than or equal to β_1.

A focal point strategy for User 2 is similarly defined. In the above definition, it is understood that the entries of the strategy m-tuple add to the throughput demand.

It is obvious from the above that not every combination of focal point strategies of the two users will result in a Nash equilibrium strategy. There will, however, be a good number of focal point strategies which will also be Nash equilibrium strategies as discussed in the following subsection.

We shall call such focal point strategies which are also Nash equilibrium strategies focal Nash equilibrium strategies (FNEs).

Another question is about the efficiency of such focal Nash equilibrium strategies. To this end, we recollect the concept of a Pareto optimal solution. It is a strategy set corresponding to the state of a game where resources are allocated in the most efficient manner. In other words, Pareto optimality is a set of conditions under which the state of economic efficiency (where no one can be made better off without making someone worse off) occurs. Pareto optimality thus corresponds to the situation or strategies where no player can be better off without adversely affecting some other player.

For the network problem under consideration, a Pareto optimal solution can be defined as a strategy set of two users for which the total combined cost of the two users is a minimum.

Thus, to the two queries raised above, we can add the third one, as follows:

Q3. From among the focal Nash equilibrium strategies, can a Pareto optimal solution be obtained? Will there be unique or many FNE's leading to the Pareto optimal solution?

To be able to answer these queries, we shall try to identify FNEs first. To this end, we need to find the relation between the numbers β_1, β_2 and λ as is done below.

4.1. Relation between $\boldsymbol{\beta_1, \beta_2}$ and $\boldsymbol{\lambda}$.

Since we are considering cycle of transmission consisting of multiple time slots, without loss of generality, we can assume that $m \geq 2$.

Case I. Consider the case $D^{(1)} = D^{(2)}$,

$$\therefore D^{(1)} = D^{(2)} = \frac{m\lambda}{2}.$$

If λ is even, say $\lambda = 2\lambda_1$, then

$$\beta_1 = \beta_2 = \left\lfloor \frac{m\lambda_1 - 1}{m} \right\rfloor = \left\lfloor \lambda_1 - \frac{1}{m} \right\rfloor = \lambda_1 - 1,$$

$$\therefore \beta_1 + \beta_2 = 2\lambda_1 - 2 = \lambda - 2. \tag{7}$$

If λ is odd, say $\lambda = 2\lambda_1 + 1$, then

$$\beta_1 = \beta_2 = \left\lfloor \frac{\frac{m}{2}(2\lambda_1+1) - 1}{m} \right\rfloor = \left\lfloor \frac{m\lambda_1 + \frac{m}{2} - 1}{m} \right\rfloor = \left\lfloor \lambda_1 + \frac{1}{2} - \frac{1}{m} \right\rfloor = \lambda_1,$$

$$\therefore \beta_1 + \beta_2 = 2\lambda_1. \tag{8}$$

That is, $\beta_1 + \beta_2 = \lambda - 1$.

Case II. Consider the case $D^{(1)} \neq D^{(2)}$.
For convenience, let $D^{(2)} < D^{(1)}$. We can write $D^{(1)} = m\lambda - D^{(2)}$.

Let
$$\beta_2 \equiv \left\lfloor \frac{D^{(2)} - 1}{m} \right\rfloor = r,$$
i.e.,
$$\frac{D^{(2)} - 1}{m} = r + s,$$
for some s, where $0 \le s < 1$,
$$\therefore\ D^{(2)} = mr + ms + 1, \tag{9}$$

$$\therefore \frac{D^{(1)} - 1}{m} = \frac{m\lambda - (mr + ms + 1) - 1}{m} = \lambda - r - \left(s + \frac{2}{m}\right),$$

$$\therefore \left\lfloor \frac{D^{(1)} - 1}{m} \right\rfloor = \left\lfloor \lambda - r - \left(s + \frac{2}{m}\right) \right\rfloor. \tag{10}$$

Obviously, $\lambda > r$ as $D^{(1)} + D^{(2)} = m\lambda$. Then

$$\therefore\ \beta_1 = \left\lfloor \frac{D^{(1)} - 1}{m} \right\rfloor = \left\lfloor \lambda - r - \left(s + \frac{2}{m}\right) \right\rfloor. \tag{11}$$

From (9), $D^{(2)} = mr + ms + 1$ yields $sm + 1 = D^{(2)} - mr$, which implies
$$\frac{sm + 2}{m} = \frac{D^{(2)} - mr + 1}{m},$$
i.e.,
$$\frac{sm + 2}{m} = \frac{D^{(2)} + 1}{m} - r.$$
Hence
$$\beta_1 = \lambda - r - 1$$
iff
$$\frac{sm + 2}{m} \le 1.$$
Therefore,
$$\frac{D^{(2)} + 1}{m} - r \le 1$$
iff
$$D^{(2)} \le m + mr - 1.$$

With $r = \beta_2$, we shall have
$$\beta_1 + \beta_2 = \lambda - 1 \quad \text{iff} \quad D^{(2)} \le m + mr - 1. \tag{12}$$

However, $D^{(2)} = mr + ms + 1,\ 0 \le s < 1$ implies $D^{(2)} < mr + m + 1$.

Thus, in general, we have
$$D^{(2)} \le mr + m. \tag{13}$$

If $D^{(2)} = m + mr$, we shall have
$$\beta_2 = \left\lfloor \frac{D^{(2)} - 1}{m} \right\rfloor = \left\lfloor \frac{m + mr - 1}{m} \right\rfloor = \left\lfloor 1 + r - \frac{1}{m} \right\rfloor = r$$
and
$$\beta_1 = \left\lfloor \frac{m\lambda - D^{(2)} - 1}{m} \right\rfloor = \left\lfloor \frac{m\lambda - m - mr - 1}{m} \right\rfloor = \left\lfloor \lambda - r - 1 - \frac{1}{m} \right\rfloor = \lambda - r - 2,$$
giving
$$\beta_1 + \beta_2 = \lambda - 2. \tag{14}$$

Thus, summarizing the two cases discussed above, we have the following: If m divides $D^{(1)}$ (and hence $D^{(2)}$), then
$$\beta_1 + \beta_2 = \lambda - 2,$$
otherwise
$$\beta_1 + \beta_2 = \lambda - 1. \tag{15}$$

In case m divides $D^{(1)}$ (and hence $D^{(2)}$), it follows that
$$\beta_1 + 1 = \frac{D^{(1)}}{m}$$
and
$$\beta_2 + 1 = \frac{D^{(2)}}{m}.$$
Hence
$$\alpha_1 = \alpha_2 = m. \tag{16}$$

4.2. Focal Nash equilibrium strategies and the Pareto optimal solution. We shall now try to identify focal Nash equilibrium strategies first and then see if some of them are also a Pareto optimal solution.

To this end, we shall use the relations (15) between β_1, β_2 and λ.

We first consider the case $\beta_1 + \beta_2 = \lambda - 1$. Here, $\alpha_1 + \alpha_2 = m$ (follows from Eqn. (6)). Therefore, a strategy set for which the m-tuple strategy of User 1 has entries $\beta_1 + 1$ in any α_1 places and β_1 in the remaining $m - \alpha_1$

places while the m-tuple strategy of User 2 has entries β_2 whenever User 1's m-tuple has entries $\beta_1 + 1$ and the remaining entries of User 2's m-tuple strategy are $\beta_2 + 1$, each will be an FNE.

In the case $\beta_1 + \beta_2 = \lambda - 2$, we have $\alpha_1 = \alpha_2 = m$, and FNEs will be the collection of all m-tuple strategy pairs of User 1 and User 2 which are such that User 1's m-tuple will have possible entries $\beta_1 + 2$ or $\beta_1 + 1$ or β_1 and corresponding entries in the m-tuple of User 2's strategy given by β_2 or $\beta_2 + 1$ or $\beta_2 + 2$, respectively.

Our next objective is to locate a Pareto optimal solution for the given problem. The best way is to search for it among the FNEs. To this end, we first identify a typical class of FNEs. As the Pareto optimal solution has been defined as the one set of strategies for which the total the cost to the two users is minimum, we use this criterion to shortlist some of the FNEs. If $D^{(1)} \geq D^{(2)}$, then from the fact that cost of transmission increases with respect to both time as well as the number of data packets, minimization of the total cost may be achieved if the cost for the first user is minimized and then, accordingly, the strategy for User 2 is determined. For the case $\beta_1 + \beta_2 = \lambda - 1$, we thus consider that FNE which has entries $\beta_1 + 1$ in the first α_1 places in the m-tuple strategy of User 1. For User 2 the strategy m-tuple will get fixed as the pair of strategies is an FNE. Similarly, for the case $\beta_1 + \beta_2 = \lambda - 2$, we consider that FNE which has all entries as $\beta_1 + 1$ in the m-tuple strategy of User 1 and entries $\beta_2 + 1$ in the m-tuple strategy of User 2.

We shall call such a strategy set a preferred focal Nash equilibrium (PFNE) strategy. Thus the pair of strategy m-tuple corresponding to the PFNE is given by

$$\{(\beta_1 + 1, \beta_1 + 1, \ldots (\alpha_1 \text{ times}), \beta_1, \beta_2, \ldots \beta_1)\},$$

$$\{(\beta_2, \beta_2, \ldots (\alpha_1 \text{times}), \beta_2 + 1, \beta_2 + 1 \ldots \beta_2 + 1)\}$$

and

$$\{(\beta_1 + 1, \beta_1 + 1, \ldots \beta_1 + 1, \beta_1 + 1)\},$$

$$\{(\beta_2 + 1, \beta_2 + 1, \ldots \beta_2 + 1, \beta_2 + 1)\},$$

respectively, for the cases $\beta_1 + \beta_2 = \lambda - 1$ and $\beta_1 + \beta_2 = \lambda - 2$.

Figure 1 shows the relation between various strategies discussed in this paper. The Pareto optimal point lies inside the set of the Nash equilibrium points and it is very difficult to achieve the Pareto optimal point by the users, so with the help of the focal Nash equilibrium and the preferred focal Nash equilibrium the user can sometimes achieve the Pareto optimal point.

Now the following questions arise:

Q.1 When does the preferred focal Nash equilibrium (PFNE) becomes the Pareto optimal solution (POS)?

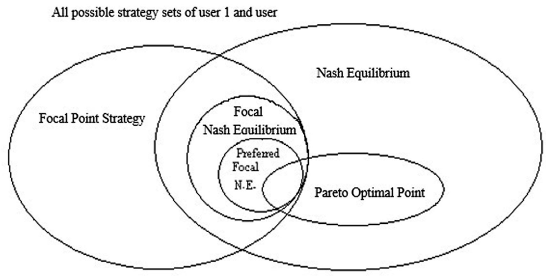

Fig. 1. Relation between various strategies.

Q.2 Is there any condition to get the POS using the PFNE?

Our next objective is to find the conditions discussed in the questions.

5. Determining conditions of the Pareto optimal solution using examples

In this section, we will determine various factors which are responsible for the POS using a Nash equilibrium point. The cost function depends on the number of data packets and it is differentiable. The second derivatives of any function reveal the nature of the extreme point, which helps us characterize the strategy. Therefore, first we check the effect of the second order derivative of cost function with respect to x_i (number of data packets).

Table 1 shows various combinations of the cost function and relations between the POS and PFNE. (In Table 1, the function $C_i^{(n)}$ is given by C.)

From Table 1, we can conclude that, if a second order derivative of the cost function with respect to x_i is zero, then the total cost for the network (i.e., the sum of the total cost incurred by both the users) remains constant for every Nash equilibrium point strategy. This constant is determined only by the cost function, m and λ. Also the probability of the PFNE to be the POS is 1. In two other cases, we mostly get a unique POS and the variable cost at different NEPs. When $\partial^2 C_i^{(n)}/\partial x_i^2 > 0$, approximately 50% PFNEs will be POSs, which is not possible in the remaining cases.

From the above study we can conclude that the nature of the second order derivative is the first and major factor for the POS.

Again we start with the case $(\partial^2 C_i^{(n)}/\partial x_i^2 > 0)$ for which approximately 50% PFNEs will be POSs. Now the question arises which condition must be satisfied for getting those 50% POSs from PFNEs. Tables 2–4 show some examples of the second case and relative study of the POS and the PFNE for different sets of m and λ.

By the observation of Tables 2–4, we conclude that, if $D^{(2)} > D^{(1)}/2$ (when $D^{(1)} \geq D^{(2)}$), then the PFNE will definitely be a POS, On the other hand if

Table 1. Effect of the second order derivative on the POS.

Functions	Nature of second order derivative	Pareto optimal solution and NEP	Probability of PENE be POS	Total cost
$C = ix$ $C = i^2x$ $C = -xi + (m+1)x + m\lambda i$ $C = i^3x$ $C = xi + 10i + 15x$	$\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} = 0$	All NEP are POS	1	Constant (its value depends on function used, m and λ)
$C = ix^2$ $C = x^3i$ $C = x^2i + 10i + 15x$	$\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} > 0$	Mostly unique POS	≈ 0.5	Variable
$C = e^i x^{1/2}$	$\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} < 0$	Unique POS	0	Variable

Table 2. Detailed study of $\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} > 0$ when $m = 4$ and $\lambda = 6$.

$D^{(1)}$	$D^{(2)}$	Pareto optimal strategy	PFNE strategy	Whether PFNE is POS
18	6	$\{(6,4,4,4),(0,2,2,2)\}$	$\{(5,5,4,4),(1,1,2,2)\}$	No
15	9	$\{(4,4,4,3),(2,2,2,3)\}$ (Not unique)	$\{(4,4,4,3),(2,2,2,3)\}$	Yes
14	10	$\{(4,4,3,3),(2,2,3,3)\}$ (unique)	$\{(4,4,3,3),(2,2,3,3)\}$	Yes
16	8	$\{(5,4,4,3),(1,2,2,3)\}$	$\{(4,4,4,4),(2,2,2,2)\}$	No
20	4	$\{(6,5,5,4),(0,1,1,2)\}$	$\{(5,5,5,5),(1,1,1,1)\}$	No

$D^{(2)} \leq D^{(1)}/2$, then it cannot be a POS. From the above discussion we come to the point that another factor which affected the POS from the PFNE is the relation between demands of both the users. In each pair of demands, the lower demand must be greater than 50% of the larger one.

6. Comparative analysis of the model

The traffic engineering (TE) method was introduced to optimize the cost and performance of traffic delivering by an online service provider (OSP) network to its users (Zhang *et al.*, 2010). The authors assumed that consistently applying low-cost strategies in each short interval can reduce the actual traffic cost over the billing period. In a similar manner we have also assumed that utilizing the complete capacity of link in each and every time slot can reduce the cost of packet transmission over the link.

As we have discussed in this paper, many strategies (depending on $m, \lambda, D^{(1)}$ and $D^{(2)}$) exist for a given combination but the Nash equilibrium strategies will be optimal. Zhang *et al.* (2010) also observed that the number of strategies is combinatorial but not all strategies are worth exploring, only a small subset of optimal strategies need to be considered. They used a linear programming problem (LPP) to minimize the pseudocost, which consists of capacity constraints, wRTT (weighted average round trip time) constraints and the constraint ensuring traffic to the destination constrains.

Similarly, throughput optimization routing and network congestion minimization routing problems are formulated in LPP (Wellons *et al.*, 2008) for predictive and oblivious mesh network routing. Instead of LPP, we have used a game theoretic model. Constraints used in our model are described in Section 2.2. To find an optimal solution of this model, we concentrate on Nash equilibrium strategies of the game. Specifically, to minimize the network cost, we found strategies which are Pareto optimal for the game. This paper presents the mathematical aspect of optimizing a network, which was described in the game theoretic view.

Table 3. Detailed study of $\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} > 0$ when $m = 4$ and $\lambda = 5$.

$D^{(1)}$	$D^{(2)}$	Pareto optimal strategy	PFNE strategy	Whether PFNE is POS
10	10	$\{(3,3,2,2),(2,2,3,3)\}$	$\{(3,3,2,2),(2,2,3,3)\}$	Yes
11	9	$\{(3,3,3,2),(2,2,2,3)\}$	$\{(3,3,3,2),(2,2,2,3)\}$	Yes
13	7	$\{(4,3,3,3),(1,2,2,2)\}$	$\{(4,3,3,3),(1,2,2,2)\}$	Yes
14	6	$\{(4,4,3,3),(1,1,2,2)\}$ $\{(5,3,3,3),(0,2,2,2)\}$	$\{(4,4,3,3),(1,1,2,2)\}$	Yes
15	5	$\{(5,4,3,3),(0,1,2,2)\}$	$\{(4,4,4,3),(1,1,1,2)\}$	No
12	8	$\{(3,3,3,3),(2,2,2,2)\}$	$\{(3,3,3,3),(2,2,2,2)\}$	Yes

Table 4. Detailed study of $\frac{\partial^2 C_i^{(n)}}{\partial x_i^2} > 0$ when $m = 4$ and $\lambda = 7$.

$D^{(1)}$	$D^{(2)}$	Pareto optimal strategy	PFNE strategy	Whether PFNE is POS
18	10	$\{(5,5,4,4),(2,2,3,3)\}$ $\{(6,4,4,4),(1,3,3,3)\}$	$\{(5,5,4,4),(2,2,3,3)\}$	Yes
17	11	$\{(5,4,4,4),(2,3,3,3)\}$	$\{(5,4,4,4),(2,3,3,3)\}$	Yes
15	13	$\{(4,4,4,3),(3,3,3,4)\}$	$\{(4,4,4,3),(3,3,3,4)\}$	Yes
14	14	$\{(4,4,3,3),(3,3,4,4)\}$	$\{(4,4,3,3),(3,3,4,4)\}$	Yes
21	7	$\{(7,5,5,4),(0,2,2,3)\}$	$\{(6,5,5,5),(1,2,2,2)\}$	No
16	12	$\{(4,4,4,4),(3,3,3,3)\}$	$\{(4,4,4,4),(3,3,3,3)\}$	Yes
20	8	$\{(6,5,5,4),(1,2,2,3)\}$ $\{(7,5,4,4),(0,2,3,3)\}$	$\{(5,5,5,5),(2,2,2,2)\}$	No
24	4	$\{(7,6,6,5),(0,1,1,2)\}$ $\{(7,7,5,5),(0,0,2,2)\}$	$\{(6,6,6,6),(1,1,1,1)\}$	No

7. Conclusion

The model described above can indeed be considered a dynamic generalization allowing transmission even in the case where the total intended transmission exceeds the link capacity, while not being just a simple generalization where the number of links is replaced by the number of time slots.

We obtained necessary and sufficient condition for the Nash equilibrium point. We developed a procedure to find the focal point and the preferred focal point from these multiple Nash equilibrium points. We determined the conditions for the PFNE as a Pareto optimal solution.

In the future, we would like to determine a function which will provide the Pareto optimal strategy directly. As mentioned in the present study, a link is used while planning to explore the Pareto optimality conditions for the network having multiple links with various capacities. The route selection part becomes very important in this scenario.

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Received: 28 May 2020
Revised: 4 October 2020
Accepted: 12 November 2020