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DIVISIBILITY OF THE SECOND–ORDER MINORS OF THE NOMINATORS BY MINIMAL DENOMINATORS OF TRANSFER MATRICES OF CYCLIC FRACTIONAL LINEAR SYSTEMS

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The divisibility of the second-order minors of the numerators of transfer matrices by their minimal denominators for cyclic fractional linear systems is analyzed. It is shown that all nonzero second-order minors of the numerators of the transfer matrices are divisible by their minimal denominators if and only if the system matrices of fractional standard and descriptor linear systems are cyclic. The theorems are illustrated by examples of fractional standard and descriptor linear systems.

Keywords: divisibility, second-order minor, transfer matrix, cyclic system, fractional system, linear system.

1. Introduction

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century (Kilbas *et al.*, 2006; Ostalczyk, 2016; Podlubny, 1999), and another one was proposed in the 20th century by Caputo (Kaczorek, 2011b; Kaczorek and Rogowski, 2015). This idea has been used by engineers for modelling various processes (Kaczorek, 2011b; 2015; Kaczorek and Rogowski, 2015; Kilbas *et al.*, 2006, Ostalczyk, 2016; Podlubny, 1999). Mathematical fundamentals of fractional calculus are given in the monographs of Kilbas *et al.* (2006), Ostalczyk (2016) and Podlubny (1999). Positive fractional linear systems were investigated by Gantmacher (1959), Kaczorek (2019; 2016; 2010; 2011a; 2012; 2011b; 2015), as well as Kaczorek and Rogowski (2015). Positive linear systems with different fractional orders were addressed by Kaczorek (2010; 2011a) and Sajewski (2017b). A solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional orders was introduced by Kaczorek (2011a), who also analyzed the stability of nonlinear fractional systems (Kaczorek, 2019; 2016). Decentralized stabilization of descriptor fractional positive continuous-time linear systems with delays was investigated by Ruszewski (2019b), and stabilization of positive descriptor fractional discrete-time linear systems with two different fractional orders by a decentralized controller was exposed by Sajewski (2017b).

In this paper the divisibility of the second-order minors of the nominators by minimal denominators of transfer matrices of cyclic fractional standard and descriptor linear system will be investigated.

The paper is organized as follows. In Section 2 some preliminaries concerning fractional linear continuous-time systems and their transfer matrices are recalled. The main results of the paper are presented for standard fractional linear systems and descriptor fractional linear systems in Sections 3 and 4, respectively. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{n\times m}$ means the set of $n\times m$ real matrices, A^T denotes the transpose of the matrix A, M_n signifies the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n\times n$ identity matrix.

2. Fractional positive continuous-time linear systems

The following Caputo definition of the fractional derivative of α order will be used (Kaczorek, 2012; 2015; Kaczorek and Rogowski, 2015; Kilbas *et al.*, 2006;

Ostalczyk, 2016; Podlubny, 1999):

$$ {}_0D_t^\alpha f(t) = \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau, \quad 0<\alpha<1, \tag{1} $$

where

$$ \dot f(\tau) = \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}, $$

$$ \Gamma(z) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t, $$

$\Gamma(x) > 0$ is the Euler gamma function.

Consider the fractional continuous-time linear system

$$ \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \tag{2a} $$

$$ y(t) = Cx(t), \tag{2b} $$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Definition 1. (*Kaczorek, 2012; 2011b; 2015*) The fractional system (2) is called (internally) *positive* if $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^p_+$, $t \geq 0$ for any initial conditions $x(0) \in \mathbb{R}^n_+$ and all inputs $u(t) \in \mathbb{R}^m_+$, $t \geq 0$.

Theorem 1. (Kaczorek, 2012; 2011b; 2015) *The fractional system (2) is positive if and only if*

$$ A \in M_n, \quad B \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+. \tag{3} $$

The fractional positive linear system (2) is called asymptotically stable (and the matrix *A* Hurwitz) if

$$ \lim_{t\to\infty} x(t) = 0, \quad \forall x(0) \in \mathbb{R}^n_+. \tag{4} $$

The positive fractional system (2) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix *A* are negative, i.e., $\operatorname{Re} s_k < 0$ for $k = 1,\dots,n$ (Kaczorek, 2011b; Kaczorek and Rogowski, 2015).

Theorem 2. (Kaczorek, 2012; 2011b; 2015) *The positive fractional system (2) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*
(i) All coefficients of the characteristic polynomial

$$ \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{5} $$

are positive, i.e., $a_i > 0$ *for* $i = 0,1,\dots,n-1$.
(ii) There exists a strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1,\dots,n$ *such that*

$$ A\lambda < 0 \quad \textit{or} \quad \lambda^T A < 0. \tag{6} $$

The transfer matrix of the system (2) is given by

$$ T(s^\alpha) = C[I_n s^\alpha - A]^{-1}B. \tag{7} $$

Theorem 3. *For the linear positive system (2), if the matrix* $A \in M_n$ *is Hurwitz and* $B \in \mathbb{R}^{n\times m}_+$, $C \in \mathbb{R}^{p\times n}_+$, $D \in \mathbb{R}^{p\times m}_+$, *then all coefficients of the transfer matrix (7) are positive.*

The proof is similar to that for standard positive linear systems (Kaczorek, 2011b; Kaczorek and Rogowski, 2015).

It is well known (Gantmacher, 1959) that the minimal polynomial $\psi(s)$ of the matrix $A \in \mathbb{R}^{n\times n}$ is related to its characteristic polynomial

$$ \varphi(s) = \det(I_n s - A) \tag{8} $$

as follows:

$$ \psi(s) = \frac{\varphi(s)}{D_{n-1}(s)}, \tag{9} $$

where $D_{n-1}(s)$ is the greatest common divisor of all the $(n-1)$-th order minors of $[I_n s - A]$. From (9) it follows that $\psi(s) = \varphi(s)$ if and only if

$$ D_1(s) = D_2(s) = \cdots = D_{n-1}(s) = 1. \tag{10} $$

Definition 2. The matrix $A \in \mathbb{R}^{n\times n}$ satisfying the condition (10) is called the *cyclic matrix*.

The inverse matrix $[I_n s - A]^{-1}$ is a rational matrix in the variable s and it can be written in the form

$$ [I_n s - A]^{-1} = \frac{N(s)}{d(s)}, \tag{11} $$

where $N(s) \in \mathbb{R}^{n\times n}[s]$ (the set of polynomial matrices in s) and $d(s)$ is the least common denominator.

Theorem 4. (Gantmacher, 1959) *Let* $A \in \mathbb{R}^{n\times n}$ *and* $n \geq 2$. *Then every nonzero second-order minor of the polynomial matrix* $N(s) \in \mathbb{R}^{n\times n}[s]$ *is divisible without remainder by the polynomial* $d(s)$ *if and only if* $\varphi(s) = \psi(s)$.

3. Main result

The transfer matrix (7) can be written in the form

$$ T(s_\alpha) = \frac{\bar N(s_\alpha)}{d(s_\alpha)}, \quad p,\ m \geq 2, \quad s_\alpha = s^\alpha, \tag{12} $$

where $\bar N(s_\alpha) = CN(s)B \in \mathbb{R}^{p\times m}$ and $d(s_\alpha)$ is the least common denominator.

Theorem 5. *Every nonzero second-order minor of the polynomial matrix* $\bar N(s_\alpha)$ *of (12) is divisible without remainder by* $d(s_\alpha)$ *if and only if the matrix A is cyclic.*

Proof.
(Sufficiency) If the matrix A is cyclic then $\psi(s) = d(s)$ and the Smith form of $[I_n s - A]_S$ is given by

$$[I_n s - A]_S = \text{diag}[1,\ 1, \ldots,\ 1,\ d(s)]. \quad (13)$$

The adjoint matrix of the matrix (13) has the form

$$\begin{aligned}\text{Adj}[I_n s - A]_S &= N(s) \\ &= \text{diag}[d(s),\ d(s), \ldots,\ d(s),\ 1]. \quad (14)\end{aligned}$$

From (14) it follows that every nonzero second-order minor of (14) is divisible without remainder by the polynomial $d(s)$. According to the Binet–Cachy theorem (Gantmacher, 1959), every second-order minor of the matrix

$$\begin{aligned}N(s) &= \text{Adj}\,[U(s)[I_n s - A]V(s)] \\ &= V(s)\text{Adj}[I_n s - A]_S U(s)\end{aligned}$$

is the sum of the products of the second-order nonzero minors of (14) and of unimodular matrices $U(s)$ and $V(s)$. Therefore, every nonzero second-order minor of $N(s)$ is divisible without remainder by the polynomial $d(s)$.

(Necessity) From the definition of the standard form it follows that $\bar{N}(s)/d(s)$ is an irreducible fraction and the polynomial $d(s)$ is monic. If the matrix A is not cyclic then $\psi(s) \neq \varphi(s)$ and $D_{n-1}(s) \neq 1$. In this case $\bar{N}(s)/d(s)$ is not an irreducible fraction. ■

Definition 3. The fractional linear system with the transfer matrix (12) is called *normal* if every nonzero second-order minor of the polynomial matrix $N(s)$ is divisible without remainder by the polynomial $d(s)$.

Example 1. Consider the fractional linear system (2) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (15)$$

$$C = \begin{bmatrix} c_1 & 0 & c_3 \\ 0 & c_2 & 0 \end{bmatrix}.$$

Note that the matrix A has Frobenius form and is cyclic. In this case we have

$$\begin{aligned}d(s_\alpha) &= \det[I_3 s_\alpha - A] \\ &= \begin{vmatrix} s_\alpha & -1 & 0 \\ 0 & s_\alpha & -1 \\ a_0 & a_1 & s_\alpha + a_2 \end{vmatrix} \quad (16) \\ &= s_\alpha^3 + a_2 s_\alpha^2 + a_1 s_\alpha + a_0 \quad (s_\alpha = s^\alpha)\end{aligned}$$

and

$$[I_3 s_\alpha - A]^{-1} = \frac{N(s_\alpha)}{d(s_\alpha)}, \quad (17)$$

where

$$N(s_\alpha) = \begin{bmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & s_\alpha + a_2 & 1 \\ -a_0 & s_\alpha^2 + a_2 s_\alpha & s_\alpha \\ -a_0 s_\alpha & -a_1 s_\alpha - a_0 & s_\alpha^2 \end{bmatrix}. \quad (18)$$

The second-order minor of the matrix (18) are

$$\begin{aligned}
M_{11}(s_\alpha) &= \begin{vmatrix} s_\alpha^2 + a_2 s_\alpha & s_\alpha \\ -a_1 s_\alpha - a_0 & s_\alpha^2 \end{vmatrix} \\
&= s_\alpha d(s_\alpha), \\
M_{12}(s_\alpha) &= \begin{vmatrix} -a_0 & s_\alpha \\ -a_0 s_\alpha & s_\alpha^2 \end{vmatrix} = 0, \\
M_{13}(s_\alpha) &= \begin{vmatrix} -a_0 & s_\alpha^2 + a_2 s_\alpha \\ -a_0 s_\alpha & -a_1 s_\alpha - a_0 \end{vmatrix} \\
&= a_0 d(s_\alpha), \\
M_{21}(s_\alpha) &= \begin{vmatrix} s_\alpha + a_2 & 1 \\ -a_1 s_\alpha - a_0 & s_\alpha^2 \end{vmatrix} = d(s_\alpha), \\
M_{22}(s_\alpha) &= \begin{vmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & 1 \\ -a_0 s_\alpha & s_\alpha^2 \end{vmatrix} = s_\alpha d(s_\alpha), \quad (19) \\
M_{23}(s_\alpha) &= \begin{vmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & s_\alpha + a_2 \\ -a_0 s_\alpha & -a_1 s_\alpha - a_0 \end{vmatrix} \\
&= -a_1 d(s_\alpha), \\
M_{31}(s_\alpha) &= \begin{vmatrix} s_\alpha + a_2 & 1 \\ s_\alpha^2 + a_2 s_\alpha & s_\alpha \end{vmatrix} = 0, \\
M_{32}(s_\alpha) &= \begin{vmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & 1 \\ -a_0 & s_\alpha \end{vmatrix} = d(s_\alpha), \\
M_{33}(s_\alpha) &= \begin{vmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & s_\alpha + a_2 \\ -a_0 & s_\alpha^2 + a_2 s_\alpha \end{vmatrix} \\
&= (s_\alpha + a_2) d(s_\alpha).
\end{aligned}$$

The nonzero minors (19) are divisible by the polynomial $d(s_\alpha)$.

Using (15) and (17) and taking into account that $M_{12}(s_\alpha) = 0$ and $M_{31}(s_\alpha) = 0$, we obtain

$$\begin{aligned}
T(s_\alpha) &= C[I_3 s_\alpha - A]^{-1} B \\
&= \begin{bmatrix} c_1 & 0 & c_3 \\ 0 & c_2 & 0 \end{bmatrix} \frac{1}{d(s_\alpha)} \\
&\times \begin{bmatrix} s_\alpha^2 + a_2 s_\alpha + a_1 & s_\alpha + a_2 & 1 \\ -a_0 & s_\alpha^2 + a_2 s_\alpha & s_\alpha \\ -a_0 s_\alpha & -a_1 s_\alpha - a_0 & s_\alpha^2 \end{bmatrix} \\
&\times \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \frac{\bar{N}(s_\alpha)}{d(s_\alpha)},
\end{aligned} \quad (20)$$

where

$$\bar{N}(s_\alpha) = \begin{bmatrix} b_2c_1(s_\alpha + a_2) - b_2c_3(a_1s_\alpha + a_0) & b_1c_1(s_\alpha^2 + a_2s_\alpha + a_1) - b_1a_0c_3s_\alpha) \\ b_2c_2s_\alpha(a_2 + s_\alpha) & -a_0b_1c_2 \end{bmatrix}$$

and

$$\frac{\det \bar{N}(s_\alpha)}{d(s_\alpha)} = b_1b_2[c_2c_3a_0 - c_1c_2(s_\alpha + a_2)].$$

This confirms Theorem 5. ♦

Remark 1. The cancellation in Theorem 5 depends only on the cyclicity of the matrix A and it is independent of the controllability of the pair $(A,\ B)$ and the observability of the pair $(A,\ C)$ of the system.

Remark 2. It is well known that in the transfer matrix (7) the cancellation occurs if the pair $(A,\ B)$ is uncontrollable and/or if the pair $(A,\ C)$ is unobservable. In this example the pair $(A,\ B)$ is controllable and the pair $(A,\ C)$ is observable since

$$\operatorname{rank}[\ I_3s_\alpha - A \quad B\ ] = n = 3 \tag{21a}$$

and

$$\operatorname{rank}\begin{bmatrix} I_3s_\alpha - A \\ C \end{bmatrix} = n = 3. \tag{21b}$$

Remark 3. The divisibility of the second-order minors of the numerators by nominal denominators of transfer matrices is independent of the positivity of fractional linear systems.

4. Fractional descriptor linear systems

Consider the fractional descriptor linear system

$$E\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \tag{22a}$$

$$y(t) = Cx(t), \tag{22b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and the fractional derivative is defined by (1).

It is assumed that $\det E = 0$ and the pencil is regular, i.e.,

$$\det[Es_\alpha - A] \neq 0. \tag{23}$$

textfor some $s \in \mathbb{C}$.

By the Weierstrass–Kronecker theorem, if the condition (23) is satisfied, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that Eqns. (22) are equivalent to the equations

$$\frac{\mathrm{d}^\alpha x_1(t)}{\mathrm{d}t^\alpha} = A_1x_1(t) + B_1u(t), \tag{24a}$$

$$N\frac{\mathrm{d}^\alpha x_2(t)}{\mathrm{d}t^\alpha} = x_2(t) + B_2u(t), \tag{24b}$$

$$y(t) = C_1x_1(t) + C_2x_2(t), \tag{24c}$$

where $A_1 \in \mathbb{R}^{n_1\times n_1}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with index q, $N^q = 0$.

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{25}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{26}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{27}$$

$$CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q^{-1}x(t), \tag{29}$$

$$x_1(t) \in \mathbb{R}^{n_1}, \quad x_2(t) \in \mathbb{R}^{n_2}, \tag{30}$$

$$n = n_1 + n_2. \tag{31}$$

The transfer matrix of the system (24)–(31) has the form

$$T(s_\alpha) = P(s_\alpha) + T_p(s_\alpha), \tag{32}$$

where

$$P(s_\alpha) = -C_2[I_{n_2} + Ns_\alpha + \cdots + N^{q-1}s_\alpha^{q-1}]B_2 \tag{33}$$

is the polynomial part and

$$\begin{aligned} T_p(s_\alpha) &= C_1[I_{n_1}s_\alpha - A_1]^{-1}B_1 \\ &= \frac{N_p(s_\alpha)}{d_p(s_\alpha)}, \\ N_p(s_\alpha) &= C_1[I_{n_1}s_\alpha - A_1]_{ad}B_1, d_p(s_\alpha) \\ &= \det[I_{n_1}s_\alpha - A_1] \end{aligned} \tag{34}$$

is the strictly proper part.

The transfer matrix (32) can be written in the form

$$T(s_\alpha) = \frac{\hat{N}(s_\alpha)}{d_p(s_\alpha)}, \tag{35}$$

where

$$\hat{N}(s_\alpha) = d_p(s_\alpha)P(s_\alpha) + N_p(s_\alpha). \tag{36}$$

Theorem 6. *Every nonzero second-order minor of the numerator (36) of the transfer matrix (35) is divisible by its denominator $d_p(s_\alpha)$ if and only if the matrix A_1 is cyclic.*

Proof. By Theorem 5 every nonzero second-order minor of the matrix $N_p(s_\alpha)$ is divisible by $d_p(s_\alpha)$ if and only if the matrix A_1 is cyclic. Let

$$\begin{aligned}&\det\begin{bmatrix} n_{11}+d_p p_{11} & n_{12}+d_p p_{12} \\ n_{21}+d_p p_{21} & n_{22}+d_p p_{22}\end{bmatrix}\\ &= n_{11}n_{22}-n_{12}n_{21}\\ &\quad + d_p(n_{11}p_{22}+n_{22}p_{11}-n_{12}p_{21}-n_{21}p_{12})\\ &\quad + d_p^2(p_{11}p_{22}-p_{12}p_{21})\end{aligned} \tag{37}$$

be any nonzero second-order minor of the matrix (36). Note that the minor (37) is divisible by the polynomial $d_p = d_p(s_\alpha)$ if and only if the matrix A_1 is cyclic. ■

By Definition 3 the fractional descriptor system with cyclic matrix A_1 is normal.

Example 2. Consider the fractional descriptor linear system (22) with matrices

$$\begin{aligned}E &= \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0\end{bmatrix},\\ A &= \begin{bmatrix} 0 & 0 & -1 & -1.5 \\ 0 & 0 & 0 & 0.25 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0\end{bmatrix},\\ B &= \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 1 \\ 2 & 0 & 0\end{bmatrix}0,\\ C &= \begin{bmatrix} 0 & -2 & 0 & 1 \\ -2 & 0 & 2 & 0\end{bmatrix}.\end{aligned} \tag{38}$$

The matrix E is singular and the condition (23) is satisfied since

$$\det[Es_\alpha - A] = \begin{vmatrix} 0 & 0 & 1 & 0.5s_\alpha+1.5 \\ 0 & 0 & 0.25s_\alpha & -0.25 \\ -0.5 & 0 & 0 & 0 \\ s_\alpha & -0.5 & 0 & 0\end{vmatrix} \neq 0. \tag{39}$$

In this case

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0\end{bmatrix},\quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0\end{bmatrix} \tag{40}$$

and

$$\begin{aligned}PEQ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix},\\ PAQ &= \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix},\\ PB &= \begin{bmatrix} B_1 \\ B_2\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2\end{bmatrix},\\ CQ^{-1} &= \begin{bmatrix} C_1 & C_2\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0\end{bmatrix},\\ &n_1 = n_2 = 2, \quad m = 3.\end{aligned} \tag{41}$$

The nilpotent index of the matrix

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$$

is $q = 2$.

The polynomial part of the transfer matrix has the form

$$\begin{aligned}P(s_\alpha) &= -C_2[I_{n_2} + Ns_\alpha]B_2\\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix} 1 & s_\alpha \\ 0 & 1\end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2\end{bmatrix}\\ &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & s_\alpha & 2s_\alpha\end{bmatrix}\end{aligned} \tag{42}$$

and the strictly proper transfer matrix is given by

$$\begin{aligned}T_p(s_\alpha) &= C_1[I_{n_1}s_\alpha - A_1]^{-1}B_1\\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} s_\alpha & -1 \\ 2 & s_\alpha+3\end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1\end{bmatrix}\\ &= \frac{1}{s_\alpha^2+3s_\alpha+2}\begin{bmatrix} s_\alpha+3 & 2 & 1 \\ -2 & 2s_\alpha & s_\alpha\end{bmatrix}.\end{aligned} \tag{43}$$

Note that the matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3\end{bmatrix}$$

is cyclic and the nonzero second-order minors

$$\begin{vmatrix} s_\alpha+3 & 2 \\ -2 & 2s_\alpha\end{vmatrix} \tag{44}$$

and

$$\begin{vmatrix} s_\alpha + 3 & 1 \\ -2 & s_\alpha \end{vmatrix}$$

of the matrix

$$\begin{aligned} N_p(s_\alpha) &= C_1[I_{n_1}s_\alpha - A_1]_{\mathrm{ad}}B_1 \\ &= \begin{bmatrix} s_\alpha + 3 & 2 & 1 \\ -2 & 2s_\alpha & s_\alpha \end{bmatrix} \end{aligned} \tag{45}$$

are divisible by the polynomial

$$d_p(s_\alpha) = \det[I_{n_1}s_\alpha - A_1] = s_\alpha^2 + 3s_\alpha + 2.$$

Using (36), (42) and (44), we obtain

$$\begin{aligned} \hat{N}(s_\alpha) &= d_p(s_\alpha)P(s_\alpha) + N_p(s_\alpha) \\ &= (s_\alpha^2 + 3s_\alpha + 2)\begin{bmatrix} 0 & 1 & 2 \\ 2 & s_\alpha & 2s_\alpha \end{bmatrix} \\ &\quad + \begin{bmatrix} s_\alpha + 3 & 2 & 1 \\ -2 & 2s_\alpha & s_\alpha \end{bmatrix} \\ &= \begin{bmatrix} s_\alpha + 3 & s_\alpha^2 + 3s_\alpha + 4 & 2s_\alpha^2 + 6s_\alpha + 5 \\ s_\alpha^2 + 6s_\alpha + 2 & s_\alpha^3 + 3s_\alpha^2 + 4s_\alpha & 2s_\alpha^3 + 6s_\alpha^2 + 5s_\alpha \end{bmatrix} \end{aligned} \tag{46}$$

It is easy to check that the nonzero second-order minors of (46) of the forms

$$\begin{aligned} &\begin{vmatrix} s_\alpha + 3 & s_\alpha^2 + 3s_\alpha + 4 \\ 2s_\alpha^2 + 6s_\alpha + 2 & s_\alpha^3 + 3s_\alpha^2 + 4s_\alpha \end{vmatrix}, \\ &\begin{vmatrix} s_\alpha + 3 & 2s_\alpha^2 + 6s_\alpha + 5 \\ 2s_\alpha^2 + 6s_\alpha + 2 & 2s_\alpha^3 + 6s_\alpha^2 + 5s_\alpha \end{vmatrix} \end{aligned} \tag{47}$$

are divisible by the polynomial $d_p(s_\alpha) = s_\alpha^2 + 3s_\alpha + 2$ and the minor

$$\begin{vmatrix} s_\alpha^2 + 3s_\alpha + 4 & 2s_\alpha^2 + 6s_\alpha + 5 \\ s_\alpha^3 + 3s_\alpha^2 + 4s_\alpha & 2s_\alpha^3 + 6s_\alpha^2 + 5s_\alpha \end{vmatrix} = 0.$$

This confirms Theorem 6. ♦

5. Concluding remarks

The divisibility of the second-order minors of the numerators of transfer matrices by their minimal denominators for cyclic fractional linear systems was investigated. It was shown that all nonzero second-order minors of the numerators of the transfer matrices are divisible by their minimal denominators if and only if the system matrices of fractional standard (Theorem 5) and descriptor (Theorem 6) linear systems are cyclic. The theorems were illustrated with examples of fractional standard and descriptor linear systems. The discussion can be easily extended to discrete-time linear standard and descriptor systems. An open problem is the extension of these deliberations to fractional discrete-time linear systems.

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