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A KALMAN FILTER WITH INTERMITTENT OBSERVATIONS AND RECONSTRUCTION OF DATA LOSSES

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This paper deals with the problem of joint state and unknown input estimation for stochastic discrete-time linear systems subject to intermittent unknown inputs on measurements. A Kalman filter approach is proposed for state prediction and intermittent unknown input reconstruction. The filter design is based on the minimization of the trace of the state estimation error covariance matrix under the constraint that the state prediction error is decoupled from active unknown inputs corrupting measurements at the current time. When the system is not strongly detectable, a sufficient stochastic stability condition on the mathematical expectation of the random state prediction errors covariance matrix is established in the case where the arrival binary sequences of unknown inputs follow independent random Bernoulli processes. When the intermittent unknown inputs on measurements represent intermittent observations, an illustrative example shows that the proposed filter corresponds to a Kalman filter with intermittent observations having the ability to generate a minimum variance unbiased prediction of measurement losses.

Keywords: Kalman filter, intermittent unknown inputs, linear system, intermittent observation.

1. Introduction

Since the Kalman filter (KF) was designed by Kalman (1960), it has become the basis of different systems theories (Nosrati and Shafiee, 2018; Ding and Fang, 2018; Tran *et al*., 2021). It plays an essential role in many estimation processes in a wide range of applications (Simon, 2006; Kailath *et al*., 2000; Sumithra and Vadivel, 2021).

The state filtering problem for discrete-time systems in the presence of persistent unknown inputs has drawn close attention. Friedland (1969) proposed the two-stage Kalman filter in which the state estimation and unknown input estimation are decoupled to reduce computation requirements of the augmented state filter (see Alouani *et al*., 1992; Keller and Darouach, 1997; Hsieh and Chen, 1999; Ignagni, 2000; Kim *et al*., 2006). When there is no prior information available about the unknown input, an optimal recursive state filter by Kitanidis (1987) can be applied so that the state estimation error is decoupled from unknown inputs. Another approach which consists in transforming a standard system with unknown inputs into a singular system without unknown inputs was introduced by Darouach *et al*. (1992). Other optimal filters closer to the standard Kalman filter were derived by minimizing the estimation error covariance matrix with respect to a reduced state feedback gain. This represents the degrees of freedom in the design of the unknown input Kalman filter (UIKF) as observed by Chen and Patton (1996), Darouach and Zasadzinski (1997), and Hou and Patton (1998).

There has been a considerable amount of interest in joint estimation of input and state by using the

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Kalman filter (Varshney *et al.*, 2019; Zhang and Ding, 2020). An unbiased minimum-variance filter to estimate the unknown input and the state for linear systems, when the state and input estimation are performed in two sequential and independent steps is proposed by Gillijns and De Moor (2007). Joint state and input estimation for linear discrete-time systems is of great practical importance for fault tolerant control (FTC) and fault diagnosis problems so that each component of the unknown inputs vector has been represented as actuator or component faults (Blanke *et al.*, 2006; Hmida *et al.*, 2010; Fang *et al.*, 2011).

In recent years, networks have attracted meticulous attention with the rapid development of network technologies and novel control strategies. Networked control systems (NCSs) have been used in many industrial fields such as electrical power systems, chemical industry, manufacturing industry, natural gas systems, etc. (Yuan *et al.*, 2017; Zhang *et al.*, 2017; Wang *et al.*, 2020). The NCS is an integration of actuators, sensors, and controllers that exchange data through a communication network. This may lead to packet losses, induced delays and end-to-end communication jitters. Hespanha *et al.* (2007) review several recent results on estimation, analysis, and controller synthesis for NCSs and design control systems that take into account of effects packet losses and packet delays.

Another frequently included problem is intermittent communication that may result from unreliable channels or a stochastic manner of data transmission. This may influence the performance of the NCS components. Sun and Ma (2014) explain how the measurements could be delayed or even lost due to transmission problems and signal fluctuations of the sensor and estimator communication channels. Therefore, the problem of intermittent observations must be carefully considered (Li *et al.*, 2015; Jie *et al.*, 2018).

We are mainly concerned in our work by packet losses that cause intermittent data transmission. Sinopoli *et al.* (2004) and Fletcher *et al.* (2004) studied the particular case when the Kalman filtering problem with a random loss of observations is represented by Markovian or Bernoulli processes. It was extended later to include both random delay (Schenato *et al.*, 2007; Shi *et al.*, 2009) and packet loses. Huang and Dey (2007) consider the case where the availability of observations is regulated by a Markov chain. More recently, Censi (2010) tackled the case when the arrival of observations is driven by a semi-Markov chain. Zhang *et al.* (2012) derive a suboptimal Kalman filter with intermittent observations by minimizing the mean squared estimation error and the mean square stability has been analyzed. The dual problem of state filtering with intermittent unknown inputs on state equation is studied by Keller and Sauter (2013). Instead of using the parameterized approach proposed by Darouach and Zasadzinski (1997) which requires to pre-compute off-line the structure of the state feedback gain for each combinatorial situation of the binary sequence, the intermittent unknown input decoupling constraint was parameterized by two fixed-size matrices, called the free and constrained parts of the filter gain. The constrained gain structurally dependent on the binary sequence was linked to estimator of the intermittent unknown inputs. From a two-stage optimization strategy very similar to that described by Friedland (1969), the free and constrained gains were both used to minimize the trace of the state estimation error covariance matrix and the trace of the unknown input estimation error covariance matrix.

Besides several network-induced effects, NCSs become vulnerable to cyber physical attacks incorporating cyber and physical activities into a malicious attack that can lead to serious incidents. Recently, a sharp rise in the number of cyber attacks has been reported. Consequently, many researchers have shown a great concern for the analysis of vulnerabilities of NCSs to external attacks (Wang and Yang, 2019; Chang *et al.*, 2018; Chabir *et al.*, 2018).

Attackers can generate various types of cyber attacks. They can be categorized as deception attacks which compromise the authenticity of the sensors and actuators' data by injecting false data among them and DoS attacks which affect the availability of the data. A deception attack is generated by directly modifying the control or measurement signal. It is classified into four classes: false data injection attacks (Liang *et al.*, 2015), covert attacks (De Sá *et al.*, 2017), replay attacks (Zhu and Martinez, 2013) and stealthy attacks (Rhouma *et al.*, 2015; 2018; Dán and Sandberg, 2010). In turn, DoS attacks are introduced to corrupt the sensor measurement or the control command by affecting communication channels of NCSs. These malicious acts may cause time delays and packet dropouts as described by Huang *et al.* (2011) as well as Yuan and Sun (2015). Since DoS attacks require little prior knowledge on control systems, they are easy to apply and the study of NCSs under DoS attacks becomes of paramount importance.

Motivated by the aforementioned discussions, this paper is devoted to handle the issue of joint state and unknown input estimation for stochastic discrete-time linear systems subject to intermittent unknown inputs on measurements. We propose a Kalman filter approach for state prediction and intermittent unknown input reconstruction. The filter design is based on the minimization of the trace of the state estimation error covariance matrix under the constraint that the state prediction error is decoupled from active unknown inputs corrupting measurements at the current time. When the system is not strongly detectable, this work establishes a sufficient stochastic stability condition on the mathematical expectation of the random state prediction errors covariance matrix when the arrival binary sequences of unknown inputs follow independent random Bernoulli processes. When the intermittent unknown inputs on measurements are used to represent intermittent observations, we show that the proposed filter coincides with a Kalman filter with intermittent observation having the ability the reconstruct measurement losses. We give an illustrative example which presents the state filtering results of the proposed Kalman filter for linear systems subject to DoS attacks and we show the ability of this strategy to reconstruct the measurement losses caused by these malicious acts.

The paper is organized as follows. Section 2 explains the problem of joint state and unknown input estimation with intermittent unknown inputs in measurement equation. Section 3 solves the state filtering problem and studies the stability of the stochastic filter. An illustrative example, applied to the case of DoS attacks, is given in Section 4 before conclusions in Section 5.

2. Problem formulation

Consider the following linear discrete-time stochastic systems:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1a}$$

$$y_k = Cx_k + Jd_k^\theta + v_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^d$, $y_k \in \mathbb{R}^m$and $d_k^\theta \in \mathbb{R}^q$ are the state, control, measurement and unknown input vectors with $q \leq m$. Matrices A, B, C and J are of appropriate dimensions. The process and sensor noises $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero mean uncorrelated Gaussian random sequences with

$$E\left\{\begin{bmatrix} w_k \\ v_k \end{bmatrix}\begin{bmatrix} w_j \\ v_j \end{bmatrix}^T\right\} = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}\delta_{k,j}, \tag{2}$$

where $W \geq 0$.

The initial state x_0, assumed to be uncorrelated with w_k and v_k, is a Gaussian random variable with $E\{x_0\} = \bar{x}_0$ and

$$P_0 = E\left\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right\} \geq 0.$$

The vector of intermittent unknown inputs

$$d_k^\theta = \begin{bmatrix} \rho_k^1 d_k^1 & \dots & \rho_k^i d_k^i & \dots & \rho_k^q d_k^q \end{bmatrix}^T \tag{3}$$

depends on the known binary variables

$$\theta_k = \left\{\rho_k^1, \dots, \rho_k^i, \dots, \rho_k^q\right\} \tag{4}$$

with $\rho_k^i = 1$ when the i-th component $\rho_k^i d_k^i$ of d_k^θ is active; otherwise, $\rho_k^i = 0$. Here

$$s_k = \sum_{i=1}^{q} \rho_k^i$$

represents the number of active unknown inputs. We assume that $\operatorname{rank}(J) = q \leq m$ with $J = \begin{bmatrix} j^1 & \dots & j^i & \dots & j^q \end{bmatrix}$ where j^i is the unknown input distribution vector of d_k^i.

Consider the following linear state filter:

$$\hat{x}_{k+1/k}^\theta = A\hat{x}_{k/k}^\theta + Bu_k, \tag{5a}$$

$$P_{k+1/k}^\theta = AP_{k/k}^\theta A^T + W, \tag{5b}$$

$$\hat{x}_{k/k}^\theta = \hat{x}_{k/k-1}^\theta + K_k^\theta(y_k - C\hat{x}_{k/k-1}^\theta), \tag{5c}$$

$$P_{k/k}^\theta = (I - K_k^\theta C)P_{k/k-1}^\theta(I - K_k^\theta C)^T \tag{5d}$$

$$+ K_k^\theta K_k^{\theta T}, \tag{5e}$$

where $\hat{x}_{k/k-1}^\theta$ is the state prediction with covariance matrix

$$P_{k/k-1}^\theta = E\left\{(x_k - \hat{x}_{k/k-1}^\theta)(x_k - \hat{x}_{k/k-1}^\theta)^T\right\}$$

based on measurements available until time $k-1$ and θ_{k-1} and where $\hat{x}_{k/k}^\theta$ is the state estimate with covariance matrix

$$P_{k/k}^\theta = E\left\{(x_k - \hat{x}_{k/k}^\theta)(x_k - \hat{x}_{k/k}^\theta)^T\right\}$$

based on measurements available until time k and θ_k. From (1) and (5), the state prediction error $e_{k+1/k}^\theta = x_{k+1} - \hat{x}_{k+1/k}^\theta$ and the state estimation error $e_{k/k}^\theta = x_k - \hat{x}_{k/k}^\theta$ propagate as

$$e_{k+1/k}^\theta = Ae_{k/k}^\theta + w_k, \tag{6a}$$

$$e_{k/k}^\theta = (I - K_k^\theta C)e_{k/k-1}^\theta - K_k^\theta v_k \tag{6b}$$

$$- K_k^\theta J d_k^\theta. \tag{6c}$$

Under $E\{e_{k/k-1}^\theta\} = 0$, we have $E\{e_{k/k}^\theta\} = -K_k^\theta J d_k^\theta = 0$ and thus $E\{e_{k+1/k}^\theta\} = 0$ if and only if the state feedback gain $K_k^\theta \in \mathbb{R}^{n,m}$ satisfies the unknown input decoupling constraint $K_k^\theta J d_k^\theta = 0$ so that it can be equivalently rewritten under (3) as

$$K_k^\theta J_k^\theta = 0 \tag{7}$$

with $J_k^\theta = \begin{bmatrix} \rho_k^1 j^1 & \dots & \rho_k^i j^i & \dots & \rho_k^q j^q \end{bmatrix}$. Instead of parameterizing the solution $K_k^\theta = L_k^\theta \Sigma_k^\theta$ to (7) as in the work of Darouach and Zasadzinski (1997) by one parameter $L_k^\theta \in \mathbb{R}^{n,m-s_k}$, with $\Sigma_k^\theta = \alpha_k(I - J_k^\theta(J_k^\theta)^+)$ and $\alpha_k \in \mathbb{R}^{m-s_k,m}$ so that $\operatorname{rank}(\Sigma_k^\theta) = m - s_k$, which requires pre-computing α_k for each combinatorial situation of the binary sequence θ_k, this paper parameterizes the solution $K_k^\theta = K_k^0 + \mu_k^\theta G_k^\theta$ to (7) as done by Keller and Sauter (2013) by two free parameters $K_k^0 \in \mathbb{R}^{n,m}$ and $G_k^\theta \in \mathbb{R}^{q,m}$, where $\mu_k^\theta = -K_k^0 J_k^\theta$ can be easily computed on-line. The unknown input decoupling constraint (7) can then be rewritten as

$$(K_k^0 + \mu_k^\theta G_k^\theta) J_k^\theta = 0 \tag{8}$$

or

$$\mu_k^\theta G_k^\theta J_k^\theta = \mu_k^\theta. \tag{9}$$

From (9), we deduce that the state feedback gain (8) satisfies (7) $\forall \theta_k$ and $\forall K_k^0$ if and only if G_k^θ satisfies

$$G_k^\theta J_k^\theta = I_k^\theta \tag{10}$$

with $I_k^\theta = \mathrm{diag}\begin{bmatrix}\rho_k^1 & \rho_k^i & \rho_k^q\end{bmatrix}$ since $\mu_k^\theta I_k^\theta = \mu_k^\theta$, $\forall \theta_k$ and $\forall K_k^0$. The necessary and sufficient existence condition

$$\mathrm{rank}(\begin{bmatrix} J_k^\theta \\ I_k^\theta \end{bmatrix}) = \mathrm{rank}(J_k^\theta), \quad \forall \theta_k$$

for a solution to (10) is given by $\mathrm{rank}(J) = q$. Suggested by the structure of the state feedback gain (8), let us define

$$\hat{d}_{k/k}^\theta = G_k^\theta (y_k - C\hat{x}_{k/k-1}^\theta), \tag{11a}$$

$$Q_{k/k}^\theta = E\left\{ (\hat{d}_{k/k}^\theta - d_k^\theta)(\hat{d}_{k/k}^\theta - d_k^\theta)^T \right\}, \tag{11b}$$

where $E\{\hat{d}_{k/k}^\theta\} = G_k^\theta J_k^\theta d_k^\theta = d_k^\theta$ under (7) and (10). The state estimator (5) and the intermittent unknown input estimator (11) will be designed by minimizing the trace of $P_{k/k}^\theta$ and $Q_{k/k}^\theta$ with respect to K_k^θ and G_k^θ under (7) and (10).

3. Kalman filter with intermittent unknown inputs on measurements

In this section we shall solve the problem of joint state and unknown inputs estimation, presented in Section 2, by designing a Kalman filter with intermittent unknown inputs to measurements.

Theorem 1. *The unbiased minimum variance (UMV) state estimate is generated by the following modified Kalman filter:*

$$\hat{x}_{k/k}^\theta = (I - K_k^0 C)\hat{x}_{k/k-1}^\theta + K_k^0 y_k + \mu_k^\theta \hat{d}_{k/k}^\theta, \tag{12a}$$

$$P_{k/k}^\theta = (I - K_k^0 C) P_{k/k-1}^\theta (I - K_k^0 C)^T + K_k^0 K_k^{0T} + \mu_k^\theta Q_{k/k}^\theta \mu_k^{\theta T}, \tag{12b}$$

$$K_k^0 = P_{k/k-1}^\theta C^T \left(C P_{k/k-1}^\theta C^T + I\right)^{-1}, \tag{12c}$$

$$\hat{x}_{k+1/k}^\theta = A\hat{x}_{k/k}^\theta + Bu_k, \tag{12d}$$

$$P_{k+1/k}^\theta = A P_{k/k}^\theta A^T + W. \tag{12e}$$

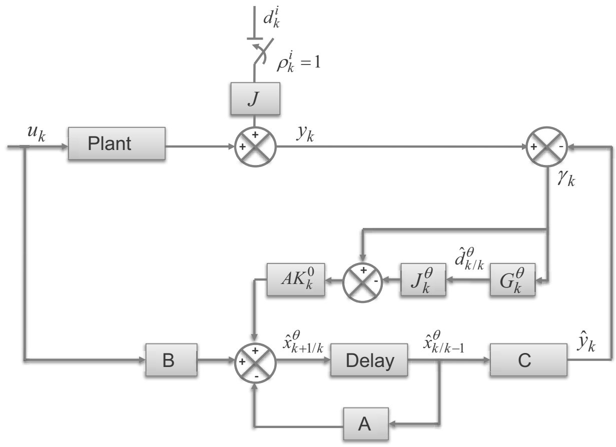

Fig. 1. Generation of $\hat{x}_{k+1/k}^\theta$ and $\hat{d}_{k/k}^\theta$ by additive correction of the standard Kalman filter.

At the estimation step, the additive quantities $\mu_k^\theta \hat{d}_{k/k}^\theta$ and $\mu_k^\theta Q_{k/k}^\theta \mu_k^{\theta T}$ depend on the unknown input estimate $\hat{d}_{k/k}^\theta$ with covariance $Q_{k/k}^\theta$ given by

$$\hat{d}_{k/k}^\theta = G_k^\theta (y_k - C\hat{x}_{k/k-1}^\theta), \tag{13a}$$

$$Q_{k/k}^\theta = [J_k^{\theta T} (C P_{k/k-1}^\theta C^T + I)^{-1} J_k^\theta]^+ \tag{13b}$$

with $G_k^\theta = Q_{k/k}^\theta J_k^{\theta T} \left(C P_{k/k-1}^\theta C^T + I\right)^{-1}$. The i-th component $\hat{d}_{k/k}^{\theta i}$ of $\hat{d}_{k/k}^\theta$ represents the estimate of $\rho_k^i d_k^i$ (with $\hat{d}_{k/k}^{\theta i} = 0$ when $\rho_k^i = 0$). On the other hand, the i-th component $Q_{k/k}^{\theta i}$ on the diagonal part of $Q_{k/k}^\theta$ represents the variance of $\hat{d}_{k/k}^{\theta i}$ (with $Q_{k/k}^{\theta i} = 0$ when $\rho_k^i = 0$). The intermittent unknown input Kalman filter with unknown inputs to measurements (IIKFM) is initialized by $\hat{x}_{0/-1}^\theta = \bar{x}_0$ and $P_{0/-1}^\theta = P_0 \geq 0$.

Figure 1 illustrates the generation concept of the estimates of $\hat{x}_{k+1/k}^\theta$ and $\hat{d}_{k/k}^\theta$ by using an additive correction of the standard Kalman filter.

Proof. We assume that

$$X_k^\theta = \begin{bmatrix} x_k^T & d_k^{\theta T} \end{bmatrix}^T,$$

$$\hat{X}_{k/k}^\theta = \begin{bmatrix} \hat{x}_{k/k}^{\theta T} & \hat{d}_{k/k}^{\theta T} \end{bmatrix}^T,$$

$$\Omega_{k/k}^\theta = E\left\{ \begin{bmatrix} e_{k/k}^\theta \\ \varepsilon_{k/k}^\theta \end{bmatrix} \begin{bmatrix} e_{k/k}^\theta \\ \varepsilon_{k/k}^\theta \end{bmatrix}^T \right\}$$

$$L_k^\theta = \begin{bmatrix} K_k^{\theta T} & G_k^{\theta T} \end{bmatrix}^T.$$

The state estimator (5) and the unknown input estimator (11) can then be jointly expressed as

$$\hat{X}^\theta_{k/k} = \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{x}^\theta_{k/k-1} + L^\theta_k (y_k - C\hat{x}^\theta_{k/k-1}), \tag{14a}$$

$$\Omega^\theta_{k/k} = \left(\begin{bmatrix} I \\ 0 \end{bmatrix} - L^\theta_k C \right) P^\theta_{k/k-1} \left(\begin{bmatrix} I \\ 0 \end{bmatrix} - L^\theta_k C \right)^T + L^\theta_k L^{\theta\,T}_k, \tag{14b}$$

$$\hat{x}^\theta_{k+1/k} = \begin{bmatrix} A & 0 \end{bmatrix} \hat{X}^\theta_{k/k} + Bu_k, \tag{14c}$$

$$P^\theta_{k+1/k} = \begin{bmatrix} A & 0 \end{bmatrix} \Omega^\theta_{k/k} \begin{bmatrix} A & 0 \end{bmatrix}^T + W. \tag{14d}$$

When $\mathrm{tr}(P^\theta_{k/k-1})$ attains a minimum, $\hat{X}^\theta_{k/k}$ is the UMV estimate of X^θ_k (and thus $\mathrm{tr}(P^\theta_{k+1/k})$ is minimum) if and only if the augmented gain $L^\theta_k = \begin{bmatrix} K^{\theta T}_k & G^{\theta T}_k \end{bmatrix}^T$ is a solution to

$$\min_{L^\theta_k} \mathrm{tr}(\Omega^\theta_{k/k}) \quad \text{subject to} \quad L^\theta_k J^\theta_k = \begin{bmatrix} 0 \\ I^\theta_k \end{bmatrix}. \tag{15}$$

The solution of (15) is difficult to obtain since K^θ_k depends on G^θ_k through $K^\theta_k = K^0_k + \mu^\theta_k G^\theta_k$. Assume that $\bar{X}^\theta_k = T_k X^\theta_k$, $\hat{\bar{X}}^\theta_{k/k} = T_k \hat{X}^\theta_{k/k}$, $\bar{\Omega}^\theta_{k/k} = T_k \Omega^\theta_{k/k} T^T_k$ and $\bar{L}^\theta_k = T_k L^\theta_k$. The matrix T_k is an arbitrary non-singular transformation matrix of appropriate dimensions. The filter (14) can then be equivalently rewritten as

$$\hat{\bar{X}}^\theta_{k/k} = T_k \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{x}^\theta_{k/k-1} + \bar{L}^\theta_k (y_k - C\hat{x}^\theta_{k/k-1}), \tag{16a}$$

$$\bar{\Omega}^\theta_{k/k} = (T_k \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{L}^\theta_k C) P^\theta_{k/k-1} (T_k \begin{bmatrix} I \\ 0 \end{bmatrix} - \bar{L}^\theta_k C)^T + \bar{L}^\theta_k \bar{L}^{\theta\,T}_k, \tag{16b}$$

$$\hat{x}^\theta_{k+1/k} = \begin{bmatrix} A & 0 \end{bmatrix} T^{-1}_k \hat{\bar{X}}^\theta_{k/k} + Bu_k, \tag{16c}$$

$$P^\theta_{k+1/k} = \begin{bmatrix} A & 0 \end{bmatrix} T^{-1}_k \bar{\Omega}^\theta_{k/k} T^{-T}_k \begin{bmatrix} A & 0 \end{bmatrix}^T + W. \tag{16d}$$

Once $\mathrm{tr}(P^\theta_{k/k-1})$ attains its minimum, $\hat{\bar{X}}^\theta_{k/k}$ is the UMV estimate of $\bar{X}^\theta_k$ (and thus $\mathrm{tr}(P^\theta_{k+1/k})$ minimum) if and only if the transformed gain $\bar{L}^\theta_k$ is a solution to

$$\min_{\bar{L}^\theta_k} \mathrm{tr}(\bar{\Omega}^\theta_{k/k}) \quad \text{subject to} \quad \bar{L}^\theta_k J^\theta_k = T_k \begin{bmatrix} 0 \\ I^\theta_k \end{bmatrix}. \tag{17}$$

With

$$T_k = \begin{bmatrix} I & -\mu^\theta_k \\ 0 & I \end{bmatrix}$$

determined so that

$$\bar{L}^\theta_k = T_k \begin{bmatrix} K^{\theta T}_k & G^{\theta T}_k \end{bmatrix}^T = \begin{bmatrix} K^{0T}_k & G^{\theta T}_k \end{bmatrix}^T,$$

where K^0_k is now decoupled from G^θ_k, we can also verify that the transformed algebraic constraints in (17) reduce to $G^\theta_k J^\theta_k = I^\theta_k$.

After straightforward manipulations, (17) becomes

$$\min_{\substack{K^0_k \\ G^\theta_k}} \mathrm{tr}(\bar{\Omega}^\theta_{k/k}) \quad \text{subject to} \quad G^\theta_k J^\theta_k = I^\theta_k \tag{18}$$

with

$$\bar{\Omega}^\theta_{k/k} = \begin{bmatrix} P^0_{k/k} & \Omega_{12} \\ \Omega_{21} & Q^\theta_{k/k} \end{bmatrix}, \tag{19}$$

where

$$\Omega_{12} = [P^\theta_{k/k-1} C^T - K^0_k (C P^\theta_{k/k-1} C^T + I)] G^{\theta T}_k,$$

$$\Omega_{21} = G^\theta_k [P^\theta_{k/k-1} C^T - K^0_k (C P^\theta_{k/k-1} C^T + I)]^T.$$

We have

$$P^0_{k/k} = (I - K^0_k C) P^\theta_{k/k-1} (I - K^0_k C)^T + K^0_k K^{0T}_k, \tag{20a}$$

$$Q^\theta_{k/k} = G^\theta_k H_k G^{\theta T}_k. \tag{20b}$$

with $H_k = C P^\theta_{k/k-1} C^T + I$.

From $\mathrm{tr}(\bar{\Omega}^\theta_{k/k}) = \mathrm{tr}(P^0_{k/k}) + \mathrm{tr}(Q^\theta_{k/k})$ which is deduced from (19), we can conclude that the global solution to (18) coincides with the local solutions of the following decoupled optimization problems:

$$\min_{K^0_k} \mathrm{tr}(P^0_{k/k}) \tag{21}$$

with

$$\min_{G^\theta_k} \mathrm{tr}(Q^\theta_{k/k}) \tag{22}$$

subject to

$$G^\theta_k J^{\theta i}_k = I^{\theta i}_k \quad \text{for} \quad i = 1, \ldots, q,$$

where $J^{\theta i}_k$ and $I^{\theta i}_k$ represent the i-th columns of J^θ_k and I^θ_k.

The unique solution to (21) coincides with the Kalman filter gain

$$K^0_k = P^\theta_{k/k-1} C^T (C P^\theta_{k/k-1} C^T + I)^{-1}.$$

The existence of the i-th constraint in (22) is conditioned by $\rho^i_k = 1$. The solution to (22) can then be derived by minimizing

$$\Phi^\theta_k = \frac{1}{2} \mathrm{tr}(G^\theta_k H_k G^{\theta T}_k) + \sum_{i=1}^{q} \lambda^{\theta i T}_k (G^\theta_k J^{\theta i}_k, -I^{\theta i}_k), \tag{23}$$

where $\lambda_k^{\theta i} \in \mathbb{R}^{q,1}$ is the Lagrange multiplier vector satisfying $\lambda_k^{\theta i} = 0$ when $\rho_k^i = 0$ and $\lambda_k^{\theta i} \neq 0$ when $\rho_k^i = 1$. The optimality conditions for Φ_k^θ are

$$\frac{\partial \Phi_k^\theta}{\partial G_k^\theta} = G_k^\theta H_k + \sum_{i=1}^{q} \lambda_k^{\theta i} (J_k^{\theta i})^T = 0, \tag{24a}$$

$$\frac{\partial \Phi_k^\theta}{\partial \lambda_k^i} = G_k^\theta J_k^{\theta i} - I_k^{\theta i} = 0 \quad \text{if } \rho_k^i = 1\,, \quad \forall i = 1, \dots, q. \tag{24b}$$

The solution

$$G_k^\theta = -\left\{ \sum_{i=1}^{q} \lambda_k^{\theta i} J_k^{\theta i T} \right\} H_k^{-1}$$

to (24a) substituted in (24b) gives

$$-\lambda_k^\theta J_k^{\theta T} H_k^{-1} J_k^\theta = I_k^\theta, \tag{25}$$

where $\lambda_k^\theta = \begin{bmatrix} \lambda_k^{\theta 1} & \lambda_k^{\theta i} & \lambda_k^{\theta q} \end{bmatrix} \in \mathbb{R}^{q,q}$ represents the Lagrange multiplier matrix. The solution to (25) expressed as $\lambda_k^\theta = -I_k^\theta [J_k^{\theta T} H_k^{-1} J_k^\theta]^+ = [J_k^{\theta T} H_k^{-1} J_k^\theta]^+$ from $J_k^\theta I_k^\theta = J_k^\theta$ and substituted in $G_k^\theta = -\lambda_k^\theta J_k^{\theta T} H_k^{-1}$ gives

$$G_k^\theta = [J_k^{\theta T} H_k^{-1} J_k^\theta]^+ J_k^{\theta T} H_k^{-1} \tag{26}$$

leading to $Q_{k/k}^\theta = G_k^\theta H_k G_k^{\theta T} = [J_k^{\theta T} H_k^{-1} J_k^\theta]^+$ via $X^+ X X^+ = X^+$ (X^+ is the unique Moore-Penrose generalized inverse of X). The optimal gain K_k^0 substituted in (19) yields the covariance

$$\bar{\Omega}_{k/k}^\theta = \operatorname{diag}\begin{bmatrix} P_{k/k}^0 & Q_{k/k}^\theta \end{bmatrix}$$

of

$$\hat{\bar{X}}_{k/k}^\theta = \begin{bmatrix} \hat{x}_{k/k}^{0T} & \hat{d}_{k/k}^{\theta T} \end{bmatrix}^T$$

where $\hat{x}_{k/k}^0 = (I - K_k^0 C)\hat{x}_{k/k-1}^\theta + K_k^0 y_k$ derives from (16a). The optimized filter (16) recovers the IIKFM of Theorem 1 via

$$T_k^{-1} = \begin{bmatrix} I & \mu_k^\theta \\ 0 & I \end{bmatrix}$$

in (16c). ■

We are now going to study the stochastic stability conditions of the IIKFM.

From $\Sigma_k^\theta = \alpha_k (I - J_k^\theta (J_k^\theta)^+)$, where α_k is a matrix of dimension $(m - s_k, m)$ so that $\operatorname{rank}(\Sigma_k^\theta) = m - s_k$, the system (1) can be transformed into a free intermittent unknown input system

$$x_{k+1} = A x_k + B u_k + w_k, \tag{27a}$$

$$y_k^\theta = C_k^\theta x_k + v_k^\theta. \tag{27b}$$

with $y_k^\theta = \Sigma_k^\theta y_k \in \mathbb{R}^{m-s_k}$, $C_k^\theta = \Sigma_k^\theta C \in v^{m-s_k,n}$ and $v_k^\theta = \Sigma_k^\theta v_k \in \mathbb{R}^{m-s_k}$.

When designed for (27), the time-varying Kalman filter

$$\hat{x}_{k+1/k}^\theta = (A - L_k^\theta C_k^\theta)\hat{x}_{k/k-1}^\theta + L_k^\theta y_k^\theta, \tag{28a}$$

$$P_{k+1/k}^\theta = (A - L_k^\theta C_k^\theta) P_{k/k-1}^\theta (A - L_k^\theta C_k^\theta)^T + L_k^\theta V_k^\theta L_k^{\theta T} + W, \tag{28b}$$

with $L_k^\theta = A P_{k/k-1}^\theta C_k^{\theta T} (C_k^\theta P_{k/k-1}^\theta C_k^\theta + V_k^\theta)^{-1} \in \mathbb{R}^{n,m-s_k}$, where $V_k^\theta = \Sigma_k^\theta \Sigma_k^{\theta T} > 0$, $\forall \theta_k = \left\{\rho_k^1, \dots, \rho_k^i, \dots, \rho_k^q\right\}$ recovers the state prediction and covariance given by the IIKFM of Theorem 1.

When the arrival binary sequence of unknown inputs follows independent random Bernoulli processes with $\lambda = \Pr[\rho_k^i = 1] \in [0 \ \ 1]$ for $i \in \{1, \dots, q\}$, let λ_c be the critical arrival rate so that

$$\lim_{k\to\infty} E\left\{P_{k+1/k}^\theta\right\} < \infty$$

when $\lambda \leq \lambda_c$ and

$$\lim_{k\to\infty} E\left\{P_{k+1/k}^\theta\right\} \to \infty$$

when $\lambda > \lambda_c$. We denote by $E\{P_{k+1/k}^\theta\}$ the mathematical expectation of the random covariance $P_{k+1/k}^\theta$ taken with respect to $\{\theta_i\}_0^k$ and λ_c is the critical unknown input occurrence rate. The sufficient conditions under which

$$\lim_{k\to\infty} E\left\{P_{k+1/k}^\theta\right\} < \infty, \quad \forall \lambda \in [0, \quad \underset{\frown}{\lambda}_c]$$

will be established with $\underset{\frown}{\lambda}_c$ as the lower bound of λ_c.

Define

$$\vartheta = \{\theta_0, \theta_1, \dots, \theta_{N-2}, \theta_{N-1}\}$$

as the set of $N = 2^q$ different binary situations of

$$\theta_k = \left\{\rho_k^1, \dots, \rho_k^i, \dots, \rho_k^q\right\}$$

described by

$$\theta_0 = \{0, 0, \dots, 0, 0\}\,,$$

$$\theta_1 = \{1, 0, 0, 0, 0\}\,,$$

$$\vdots$$

$$\theta_{N-2} = \{1, 1, 0, 1, 0\}\,,$$

$$\theta_{N-1} = \{1, 1, \dots, 1, 1\}\,.$$

Let

$$\sigma_k = \left\{\sigma_k^0, \dots, \sigma_k^j, \dots, \sigma_k^{N-1}\right\}$$

be the set of binary variables defined by $\sigma_k^j = 1$ when $\theta_k = \theta_j$ or $\sigma_k^j = 0$ when $\theta_k \neq \theta_j$.

Theorem 2. *If there exist* $L_j \in \mathbb{R}^{n,m-r_j}$ *for* $j \in \{0,\dots,N-1\}$ *and* $Y \in \mathbb{R}^{n,n}$ *with* $0 < Y \le I$ *such that*

$$\Psi_\lambda(Y, L_0, L_1, \dots, L_{N-1}) > 0 \tag{29}$$

with

$$\Psi_\lambda(Y, L_0, L_1, \dots, L_{N-1}) = \begin{bmatrix} Y & \sqrt{p_0}\Omega_0 & \sqrt{p_1}\Omega_1 & \cdots & \sqrt{p_{N-1}}\Omega_{N-1} \\ \sqrt{p_0}\Omega_0^T & Y & 0 & \cdots & 0 \\ \sqrt{p_1}\Omega_1^T & 0 & Y & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \sqrt{p_{N-1}}\Omega_{N-1}^T & 0 & 0 & \cdots & Y \end{bmatrix},$$

$\Omega_j = YA + L_jC_j$, *so that* C_j *is the value of* C_k^θ *when* $\sigma_k^j = 1$ *and* $p_j = \lambda^{r_j}(1-\lambda)^{q-r_j}$, *where* r_j *is the number of ones in* θ_j, *then*

$$\lim_{k\to\infty} E\left\{P_{k/k-1}^\theta\right\} < \infty \quad \forall\lambda \in [0\ \widehat{\underline{\lambda}}_c] \tag{30}$$

with $\widehat{\underline{\lambda}}_c$ *as the solution to the LMI feasibility problem*

$$\widehat{\underline{\lambda}}_c = \arg\left\{\max_\lambda \Psi_\lambda(Y, L_0, L_1, \dots, L_{N-1}) > 0\right\}.$$

If the system under permanent unknown inputs is strongly detectable with

$$\operatorname{rank}\begin{bmatrix} -Iz + A & 0 \\ C & J \end{bmatrix} = n + q, \ \ \forall |z| \ge 1, \tag{31}$$

then

$$\lim_{k\to\infty} E\left\{P_{k/k-1}^\theta\right\} < \infty, \quad \forall\lambda \in [0,1].$$

Proof. Relation (28b) can be expressed as a switching standard Riccati difference equation (RDE)

$$P_{k+1/k}^\theta = \sum_{j=0}^{N-1} \sigma_k^j f_j(P_{k/k-1}^\theta), \tag{32}$$

where $f_j(X)$ is the Riccati operator that can be defined as follows:

$$\begin{aligned} f_j(X) &= AXA^T + W \\ &\quad - AXC_j^T(C_jXC_j^T + V_j)^{-1}C_jXA^T \end{aligned} \tag{33}$$

and V_j is the value of V_k^θ when $\sigma_k^j = 1$. The mathematical expectation $E\{P_{k+1/k}^\theta\}$ of the random covariance $P_{k+1/k}^\theta$ can then be expressed as

$$E\left\{P_{k+1/k}^\theta\right\} = \sum_{j=0}^{N-1} p_j E\left\{f_j(P_{k/k-1}^\theta)\right\}. \tag{34}$$

The Riccati operator $f_j(X)$ is concave and increasing with X. Jensen's inequality gives

$$E\left\{P_{k+1/k}^\theta\right\} \le \sum_{j=0}^{N-1} p_j f_j(E\left\{P_{k/k-1}^\theta\right\})$$

and the deterministic upper bounded S_{k+1} of $E\{P_{k+1/k}^\theta\}$ so that $E\{P_{k+1/k}^\theta\} \le S_{k+1}$ is generated by the following modified RDE:

$$S_{k+1} = \sum_{j=0}^{N-1} p_j f_j(S_k), \tag{35}$$

with $S_0 = P_0 \ge 0$.

Define

$$S = \sum_{j=0}^{N-1} p_j f_j(S)$$

as the modified algebraic Riccati difference equation (ARDE) associated with the modified RDE (35). Theorem 2 directly yields the stochastic stability of the Kalman filter with intermittent observations. For a given λ, there exists $S \ge 0$, a solution to $S = g_\lambda(S)$ so that

$$\lim_{k\to\infty} E\left\{P_{k/k-1}^\theta\right\} \le S < \infty$$

if there exists $L_j \in \mathbb{R}^{n,m-r_j}$ for $j \in \{0,1,\dots,N-1\}$ and $0 < Y \le I$ so that $\Psi_\lambda(Y, L_0, \dots, L_{N-1}) > 0$. The solution to the LMI feasibility problem

$$\widehat{\underline{\lambda}}_c = \arg\left\{\max_\lambda \Psi_\lambda(Y, L_0, L_1, \dots, L_{N-1}) > 0\right\}$$

gives the lower bound $\widehat{\underline{\lambda}}_c$ of λ_c (Sinopoli *et al.*, 2004). In the permanent unknown inputs case, when $\lambda = 1$, the modified ARDE

$$S = \sum_{j=0}^{N-1} p_j f_j(S)$$

can be rewritten as a standard ARDE,

$$\begin{aligned} S_{k+1} &= AS_kA^T + W \\ &\quad - AS_kC_{N-1}^T(C_{N-1}S_kC_{N-1}^T + V_{N-1})^{-1} \\ &\quad \times C_{N-1}S_kA^T, \end{aligned} \tag{36}$$

with $C_{N-1} = \Sigma_{N-1}C \in \mathbb{R}^{m-q,m}$ and $V_{N-1} = \Sigma_{N-1}\Sigma_{N-1}^T \in \mathbb{R}^{m-q,m-q}$ where $\alpha \in \mathbb{R}^{m-q,m}$ in

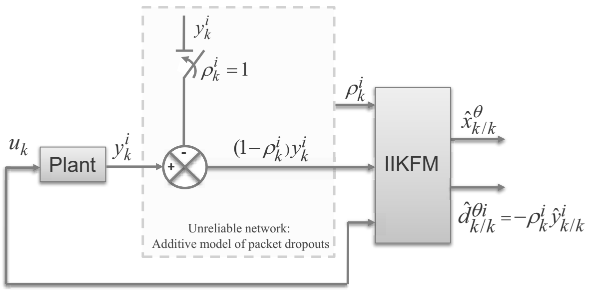

Fig. 2. IIKFM used as a KF with intermittent observations and data losses reconstruction.

$\Sigma_{N-1} = \alpha(I - JJ^+)$ is so that $\operatorname{rank}(\Sigma_{N-1}) = m - q$. Under (31), the pair (A, C_{N-1}) is detectable and there exists a strong solution to the ARDE (36) (all the modes of $A - L_{N-1}C_{N-1}$ with $L_{N-1} = ASC_{N-1}^T(C_{N-1}SC_{N-1}^T + V_{N-1})^{-1} \in \mathbb{R}^{n,m-q}$ are inside or on the unique circle). When $q = m$, the results given in this theorem remain valid with $C_{N-1} = 0$ and $L_{N-1} = 0$. ■

With $J = I$, $d_k = -y_k$ and the known binary variables in θ_k generated by communication protocols (TCP) (see, e.g., Sinopoli *et al.*, 2004), the IIKFM of Theorem 1 can be viewed as a Kalman filter with intermittent observation based on data losses reconstruction as explained in Fig. 2.

Compared with the Kalman filter with intermittent observation (28a) and (28b) which needs time-varying size matrices C_k^{θ} and V_k^{θ} computed for the 2^q binary situation of θ_k with $q = m$, the IIKFM which is computationally more efficient just updates its fixed structure from the binary sequence θ_k.

4. Illustrative example

In this section we illustrate the feasibility and effectiveness of our proposed Kalman filtering approach for state prediction and data losses reconstruction via a numerical example applied to the case of denial-of-service attacks on measurements. We assume the following matrix parameters for the linear stochastic discrete-time system (1):

$$A = \begin{bmatrix} 0.4 & 1 & 0 & 0 \\ 0 & 0.3 & 1 & 0 \\ 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$W = 0.01 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $J = \begin{bmatrix} j^1 & j^2 & j^3 \end{bmatrix}$, where

$$j^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the fault distribution vector of the intermittent unknown input $\rho_k^1 d_k^1$ affecting the first measurement,

$$j^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

the fault distribution vector of the intermittent unknown input $\rho_k^2 d_k^2$ affecting the second measurement, and

$$j^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

the fault distribution vector of the intermittent unknown input $\rho_k^3 d_k^3$ affecting the third measurement.

The plant (A, C, J) with $J = I$ and A unstable cannot be strongly detectable, which means that any unknown input observers or unknown input Kalman filters designed under permanent unknown inputs will become unstable. We use here the IIKFM of Theorem 1 as a Kalman filter with intermittent observations, where the first measurement y_k^1 is lost on unreliable communication channels with $y_k^1 = 0$ when $\rho_k^1 = 1$, where the second measurement y_k^2 is lost with $y_k^2 = 0$ when $\rho_k^2 = 1$ and where the third measurement y_k^3 is lost with $y_k^3 = 0$ when $\rho_k^3 = 1$. The binary sequence $\theta_k = \{\ \rho_k^1 \quad \rho_k^2 \quad \rho_k^3\ \}$ is known and assumed to follow independent random Bernoulli processes with $\lambda = \Pr(\rho_k^1 = 1) = \Pr(\rho_k^2 = 1) = \Pr(\rho_k^3 = 1)$.

The stabilizing controller is of the LQG type, where the standard Kalman filter is replaced by the IIKFM of Theorem 1. In order to use the IIKFM as a Kalman filter with intermittent observations caused, e.g., by random DoS attacks on measurements transmitted by the plant to the controller, its design model is modified so that $\rho_k^1 d_k^1$, $\rho_k^2 d_k^2$ and $\rho_k^3 d_k^3$ with $d_k^1 = -y_k^1$, $d_k^2 = -y_k^2$ and $d_k^3 = -y_k^3$ represent intermittent measurement losses. The IIKFM can then be viewed as a special structure of the Kalman filter with intermittent observation allowing measurement losses reconstruction.

Figures 3–14 illustrate the obtained results when the rate $\lambda = 0.5$ of DoS attacks is less than the lower bound

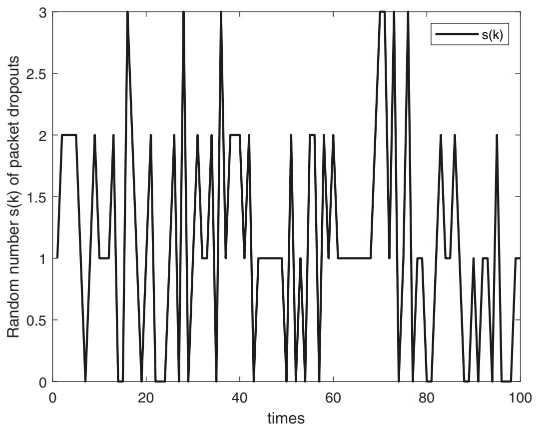

Fig. 3. Number $s(k) = \rho_k^1 + \rho_k^2 + \rho_k^3$ of DoS attacks on received measurements.

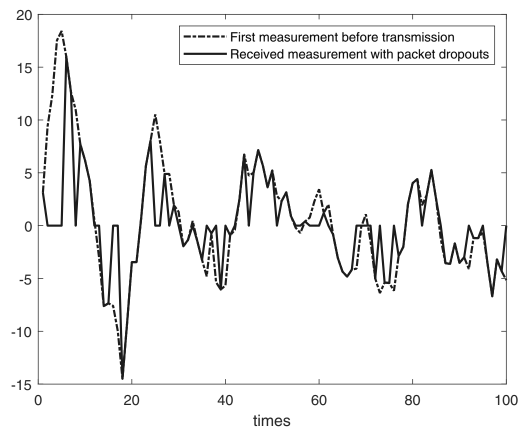

Fig. 4. First information about measurement before transmission and after reception.

$\overset{\frown}{\lambda}_c = 0,66$ of λ_c generated by the LMI of Theorem 2.
The random numbers of denial-of-service attacks are plotted in Fig. 3.

The measurements transmitted from the plant to the LQG controller via communication networks are plotted in Figs. 4–6, respectively. Information about measurements received by the LQG controller is also plotted in the same figures. Figures 7–10 present the state filtering results of the IIKFM. Figures 11–13 show the ability of the proposed filter to reconstruct the measurement losses caused by DoS attacks. Figure 14 shows that the trace of the IIKFM's state prediction error covariance matrix coincides with those given by the Kalman filter with intermittent observation.

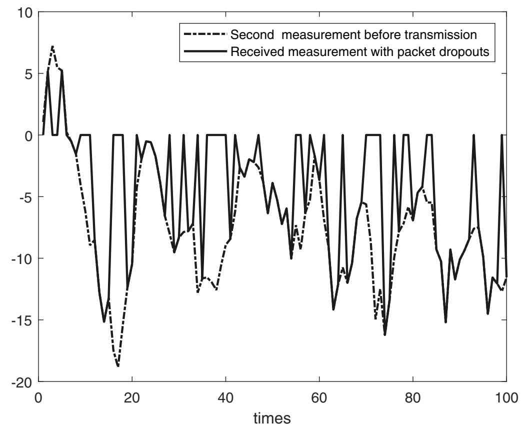

Fig. 5. Second information about measurement before transmission and after reception.

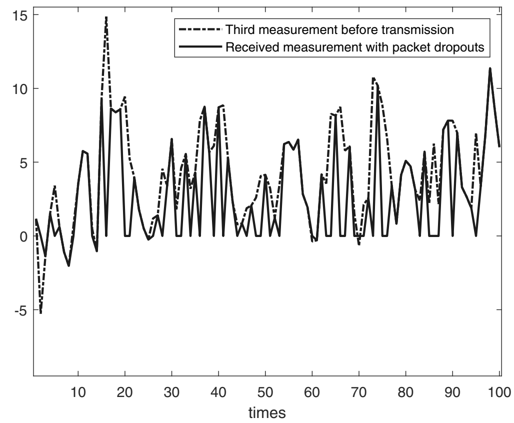

Fig. 6. Third information about measurement before transmission and after reception.

5. Conclusion

This paper has presented a Kalman filter for joint state prediction and unknown input estimation in linear stochastic discrete-time stochastic systems subject to intermittent unknown inputs to measurements. The presented linear state filter works in closed loop with a hybrid unknown input estimation and has the ability to recover the standard Kalman filter when the unknown inputs are zero. This fundamental structural property has been exploited to derive a Kalman filter with intermittent observations allowing intermittent reconstruction of measurement losses. When the system is not strongly detectable, we have established a necessary condition under which the mathematical expectation of the random state prediction errors covariance matrix is upper bounded when the arrival binary sequences of unknown inputs

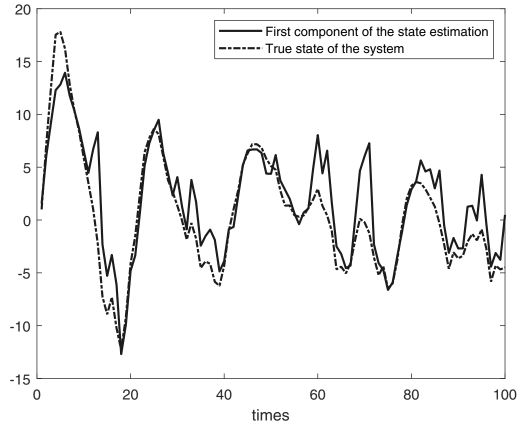

Fig. 7. First state of the plant x_k^1 and its estimate $\hat{x}_{k/k}^{\theta 1}$ generated by the IIKFM.

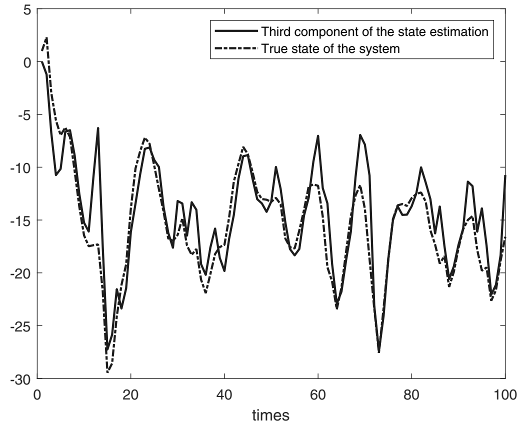

Fig. 9. Third state of the plant x_k^3 and its estimate $\hat{x}_{k/k}^{\theta 3}$ generated by the IIKFM.

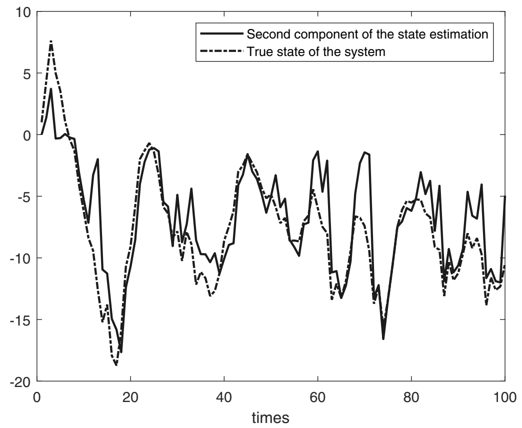

Fig. 8. Second state of the plant x_k^2 and its estimate $\hat{x}_{k/k}^{\theta 2}$ generated by the IIKFM.

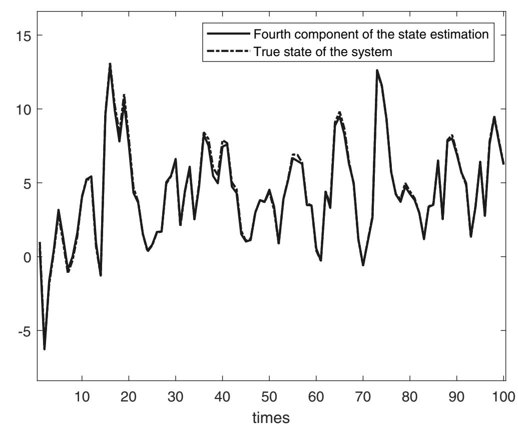

Fig. 10. Fourth state of the plant x_k^4 and its estimate $\hat{x}_{k/k}^{\theta 4}$ generated by the IIKFM.

follow independent random Bernoulli processes.

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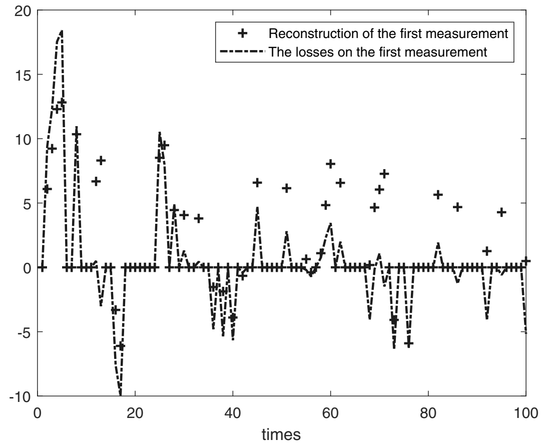

Fig. 11. Reconstruction of y_k^1 when $\rho_k^1 = 1$.

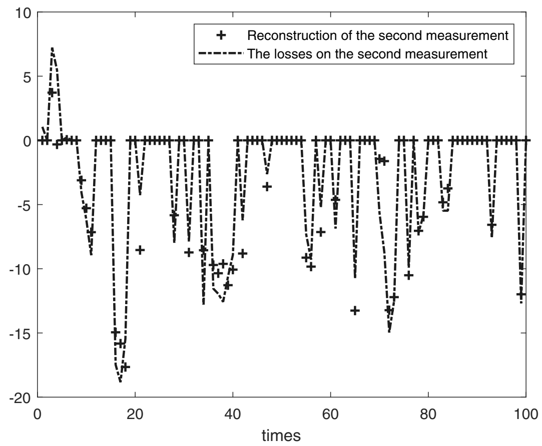

Fig. 12. Reconstruction of y_k^2 when $\rho_k^2 = 1$.

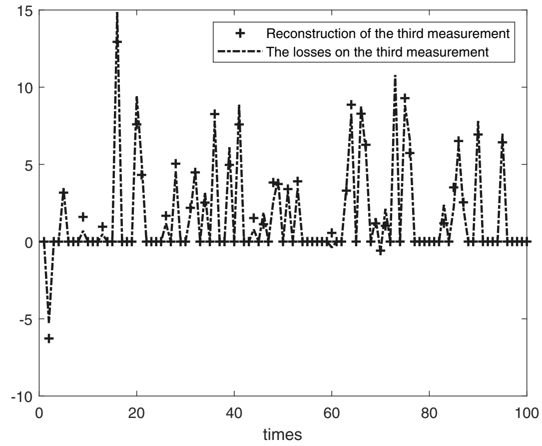

Fig. 13. Reconstruction of y_k^3 when $\rho_k^3 = 1$.

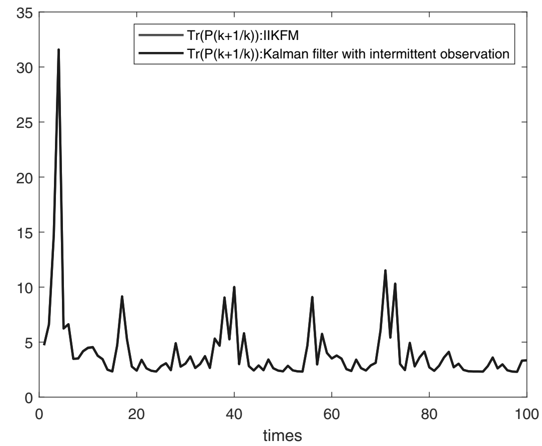

Fig. 14. Evolution of $\mathrm{tr}(P^{\theta}_{k+1/k})$ given by the IIKFM and $\mathrm{tr}(P_{k+1/k})$ given by the Kalman filter with intermittent observation.

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