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A REALISTIC TOLERANT SOLUTION OF A SYSTEM OF INTERVAL LINEAR EQUATIONS WITH THE USE OF MULTIDIMENSIONAL INTERVAL ARITHMETIC

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The paper presents a method of determining the robustness of solutions of systems of interval linear equations (ILEs). The method can be applied also for the ILE systems for which it has been impossible to find solutions so far or for which solutions in the form of improper intervals have been obtained (which cannot be implemented in practice). The research conducted by the authors has shown that for many problems it is impossible to arrive at ideal solutions that would be fully robust to data uncertainty. However, partially robust solutions can be obtained, and those with the highest robustness can be selected and put into practice. The paper shows that the degree of robustness to the uncertainty of the entire system can be calculated on the basis of the degrees of robustness of individual equations, which greatly simplifies calculations. The presented method is illustrated with a series of examples (also benchmark ones) that facilitate its understanding. It is an extension of the authors' previously published method for first-order ILEs.

Keywords: interval arithmetic, interval linear equation system, tolerable solution, multidimensional interval arithmetic.

1. Introduction

The motivation behind this paper is to show that some difficult cases of linear interval equations (ILEs) that so far have not been solved with one-dimensional interval arithmetic (1D-IA) can be solved using multidimensional interval arithmetic (MIA), and that solutions provided by MIA are more informative and more realistic (practical). The paper will consider determining robust solutions of static systems of interval equations with control variables (x_1, x_2), which can also be called decision variables. The system of ILEs describing the dependence existing in a real system can have the form

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 = b_1, \\ a_{21}\, x_1 + a_{22}\, x_2 = b_2. \end{aligned} \tag{1}$$

These types of ILE systems describe the balance relationships existing in many physical, economic, biological and other systems. Examples include the Leontief economic model (Dymova, 2011) or the patient-recommended multi-component diet model. In the case of many real systems, we do not have exact knowledge about the values of the coefficients a_{ij} occurring in the formula (1). We usually have only approximate interval knowledge of the form

$$a_{11} \in [\underline{a}_{11}, \overline{a}_{11}], \quad \ldots, \quad a_{22} \in [\underline{a}_{22}, \overline{a}_{22}]. \tag{2}$$

In the system control task (1), one should choose such values of the control variables (x_1, x_2) that in the ideal case would ensure equality of the right and left-hand sides of (1). However, because we do not know the exact values of the coefficients a_{ij} and we only have an approximate interval knowledge (2), this equality cannot be obtained in practice. Therefore, tolerance corridors $[\underline{b}_1, \overline{b}_1]$ and $[\underline{b}_2, \overline{b}_2]$ should be introduced for the output variables b_1, b_2. These tolerances facilitate (but do not always ensure) finding the appropriate control values (x_1, x_2). Under conditions of uncertain knowledge, we must therefore solve the following problem and not (1):

$$\begin{aligned} [\underline{a}_{11}, \overline{a}_{11}]\, x_1 + [\underline{a}_{12}, \overline{a}_{12}]\, x_2 = [\underline{b}_1, \overline{b}_1], \\ [\underline{a}_{21}, \overline{a}_{21}]\, x_1 + [\underline{a}_{22}, \overline{a}_{22}]\, x_2 = [\underline{b}_2, \overline{b}_2]. \end{aligned} \tag{3}$$

It should be noted that, while the uncertainties $[\underline{a}_{ij}, \overline{a}_{ij}]$ have a negative effect on the control process

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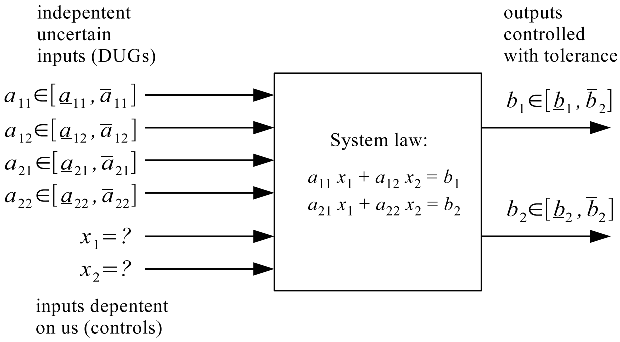

Fig. 1. Illustration of a system with two control inputs x_1, x_2 and four independent, uncertain inputs a_{11}, a_{12}, a_{21}, a_{22}.

and can be called disturbances or uncertainty generators (DUGs—disturbance uncertainty generators), which we have no influence on, the uncertainties $[\underline{b}_i,\overline{b}_i]$ in (3) can be often chosen by us depending on the application. Sometimes, however, they are also subject to the imposed accuracy requirements of the controlled system. The larger the span $[\underline{b}_i,\overline{b}_i]$, the easier it is to 'hit' tolerance corridors (TCs) and achieve the control target. There is an analogy here with shooting a target with a rifle. The described task is illustrated in Fig. 1.

The concept of a tolerable solution was formulated by S.P. Shary in the 1990s and presented in many publications (e.g., Shary, 1991; 1992; 1994; 1995). Let us consider the interval system of linear algebraic equations:

$$\mathbf{A}\mathbf{x}=\mathbf{b}\,, \tag{4}$$

where $\mathbf{A}$ is an interval $m\times n$ matrix and $\mathbf{b}$ is an m-element vector. Then the tolerable solution set can be defined as

$$\textstyle\sum_{\forall\exists}(\mathbf{A},\mathbf{b})=\{x\in\mathbb{R}^n\mid(\forall A\in\mathbf{A})(\exists b\in\mathbf{b})(Ax=b)\}\,, \tag{5}$$

or, in a shorter form, as

$$\textstyle\sum_{\forall\exists}(\mathbf{A},\mathbf{b})=\{x\in\mathbb{R}^n\mid(\forall A\in\mathbf{A})(Ax=b)\}\,. \tag{6}$$

Another type of solution that can be defined is the united solution set

$$\textstyle\sum_{\exists\exists}(\mathbf{A},\mathbf{b})=\{x\in\mathbb{R}^n\mid(\exists A\in\mathbf{A})(\exists b\in\mathbf{b})(Ax=b)\}\,. \tag{7}$$

Lodwick and Dubois (2015) define a tolerance solution set in a different way,

$$[\mathbf{A}]\mathbf{x}\in\mathbf{b}. \tag{8}$$

In the case of 1-variable ILEs, where $[\mathbf{A}]=[a]=[\underline{a},\overline{a}]$ and $\mathbf{b}=[b]=[\underline{b},\overline{b}]$, the tolerance solution $\Omega_{\forall\exists}$ is given by

$$\Omega_{\forall\exists}=\{x\mid[\underline{a},\overline{a}]x\subseteq[\underline{b},\overline{b}]\} \tag{9}$$

and supplemented by

$$\underline{b}\le ax\le\overline{b},\quad\forall a\in[\underline{a},\overline{a}]. \tag{10}$$

Both the definition (5) of the tolerable solution set (TSS) proposed by Shary (1991) and the definition (9) of Locwick and Dubois (2015) refer to a set of vectors $x\in\mathbb{R}^n$ that have full robustness to the entire set of possible matrices $A\in\mathbf{A}$. Meanwhile, in addition to vectors x that are fully robust, there may be vectors $x\in\mathbb{R}^n$ that are not fully but only partially robust. In other words, these vectors will be robust to the part $A_{hit}\subset\mathbf{A}$ of the set $\mathbf{A}$. The part A_{hit} can be large, e.g., it can constitute 95% of the set $\mathbf{A}$. Should such a vector x not be considered at all and discarded? After all, it gives a good chance of hitting the required tolerance ranges defined by the vector $\mathbf{b}$. Besides, any possibility, even a small one, of hitting the tolerance interval $\mathbf{b}$ can be valuable in a practical problem.

The novelty of the approach presented in this article is the introduction of new concepts of sets: a fully robust part of the united solution set ($USS_{FRP}=\sum_{\exists\exists FRP}(\mathbf{A},\mathbf{b})$) and a partly robust part of the united solution set ($USS_{PRP}=\sum_{\exists\exists PRP}(\mathbf{A},\mathbf{b})$). These sets are defined by the formulas (11) and (12), where $r(x)\in(0,1]$ is the degree of robustness of a single vector x to system uncertainty:

$$\begin{aligned}USS_{FRP}&=\textstyle\sum_{\exists\exists FRP}(\mathbf{A},\mathbf{b})\\&=\{x\in\mathbb{R}^n\mid(\exists A\in\mathbf{A})\\&\qquad(\exists b\in\mathbf{b})(Ax=b)(r(x)=1)\},\end{aligned} \tag{11}$$

$$\begin{aligned}USS_{PRP}&=\textstyle\sum_{\exists\exists PRP}(\mathbf{A},\mathbf{b})\\&=\{x\in\mathbb{R}^n\mid(\exists A\in\mathbf{A})\\&\qquad(\exists b\in\mathbf{b})(Ax=b)(r(x)\in(0,1))\}.\end{aligned} \tag{12}$$

It should be noted that the sum of a set of partial and full robustness is a united solution set,

$$USS_{FRP}+USS_{PRP}=USS,$$

$$\textstyle\sum_{\exists\exists FRP}(\mathbf{A},\mathbf{b})+\sum_{\exists\exists PRP}(\mathbf{A},\mathbf{b})=\sum_{\exists\exists}(\mathbf{A},\mathbf{b}). \tag{13}$$

In real-world problems, it often happens that the subset of full robustness is empty $USS_{FRP}=\emptyset$, i.e., there is no control vector $x\in\mathbb{R}^n$ that satisfies all the tolerance requirements specified by the vector $\mathbf{b}$ on the right-hand side of the equation. Then there is only a partial robust part of the USS:

$$USS=USS_{PRP},\quad\textstyle\sum_{\exists\exists}(\mathbf{A},\mathbf{b})=\sum_{\exists\exists PRP}(\mathbf{A},\mathbf{b}). \tag{14}$$

If there are both parts, USS_{FRP} and USS_{PRP}, the optimal control vector x is one of the vectors contained in USS_{FRP}, since all the vectors $x\in\mathbb{R}^n$ contained in this part have full robustness $r(x)=1$ and guarantee hitting the tolerance intervals $\mathbf{b}$. If there is only a subset USS_{PRP}, then vectors $x \in \mathbb{R}^n$ included in it have diversified robustness $0 < r(x) < 1$. Then the vector x_{opt} with the highest robustness to system uncertainty should be determined and applied to the control process. In the following, a general method of determining the robustness $r(x)$ of any single control vector $x \in \mathbb{R}^n$ will be presented.

If a linear system defined by the equation $\mathbf{A}x = \mathbf{b}$ is given, then, in the case of a continuous system, the cardinality measure $Mcard(\mathbf{A})$ of the matrix $\mathbf{A}$ of uncertain coefficients and the cardinality measure of a set $\mathbf{A}_{hit}(x) \subset \mathbf{A}$ must be determined. The matrix $\mathbf{A}_{hit}(x)$ is dependent on a particular control vector $x \in \mathbb{R}^n$. Therefore, for each vector x, it must be defined separately. The matrix $\mathbf{A}_{hit}(x)$ is defined by

$$\mathbf{A}_{hit}(x) = \{A \in \mathbf{A} \mid Ax = b \in \mathbf{b}\}. \tag{15}$$

If the cardinality measures $Mcard(\mathbf{A})$ and $Mcard(\mathbf{A}_{hit}(x))$ are specified, then the robustness $r(x)$ of the control vector $x \in \mathbb{R}^n$ can be calculated from the formula

$$r(x) = \frac{Mcard(\mathbf{A}_{hit}(x))}{Mcard(\mathbf{A})}, \qquad \mathbf{A}_{hit}(x) \subset \mathbf{A}, \quad r(x) \in [0, 1]. \tag{16}$$

The exact method of determining $Mcard(\mathbf{A})$ and $Mcard(\mathbf{A}_{hit}(x))$ for the 1st and the 2nd order linear system will be presented further on. For 2nd and higher order linear systems, due to the multidimensionality of the problem, discretization can be used. Then it is possible to determine $card(\mathbf{A})$ and $card(\mathbf{A}_{hit}(x))$ (the quantity of discrete sets), and the robustness of $r(x)$ can be calculated from the formula

$$r(x) = \frac{card(\mathbf{A}_{hit}(x))}{card(\mathbf{A})}, \quad \mathbf{A}_{hit}(x) \subset \mathbf{A}, \quad r(x) \in [0, 1]. \tag{17}$$

Both in the case of the formulas of Shary (4)–(8) as well as Lodwick Dubois (8)–(10), the elements of the solution vector $\mathbf{x}$ are intervals, i.e., 1-dimensional mathematical objects. This is due to the fact that most IA types assume that the results of calculations on intervals are also intervals. Currently, there are the following 1-dimensional IA types, quoted according to Boukezzoula *et al.* (2019) (but this is an incomplete list): standard IA (SIA), extended (generalized) IA of Kaucher, non-standard (inner) IA of Markov, generalized Hukuhara IA of Dimitrova and Stefanini, optimistic IA of Boukezzoula and Galichet, instantiated IA of Dubois (Lodwick and Dubois, 2015), constrained IA of Lodwick, single-level constrained IA of Klir, gradual IA of Dubois, Prade, Fortin, Boukezzoula (Boukezzoula *et al.*, 2014; 2019; Dubois and Prade, 2008; Fortin *et al.*, 2008).

Research into the development of new 1D-versions of IA continues. Scientists are aware of the shortcomings of the existing types of 1D-IA and are trying to develop new, more perfect versions. For example, Siahlooei and Shahzadeh Fazeli (2018) propose IA based on new inverse operations of addition and multiplication and a new concept of the general closed interval.

The large number of existing types of IA demonstrates the great difficulty in solving uncertainty problems. This difficulty has already been noticed by many scientists. For example, Dymova (2011) describes the phenomenon where some IA types give different solutions to the problem under consideration depending on the mathematical form of its presentation. Changing the form changes the calculation result. Mazandarani *et al.* (2018) described this phenomenon as unacceptable and called it unnatural behavior in modeling (UBM). The weaknesses of the current IA were also noticed by Kreinovich (2016). He emphasizes the need for a deep understanding of the interval problem before starting to solve it. Incomplete understanding of the problem often leads to inaccurate or completely incorrect results. The importance of correct interval problem solving is even greater since IA is the basic arithmetic for fuzzy arithmetic (FA). This is due to the possibility of presenting the fuzzy number as a set of α-cuts which are intervals, in accordance with the definition of Zadeh (1975). Hence, all the imperfections of IA affect the imperfections of FA. The fact that the current IA is imperfect is evidenced not only by the UBM phenomenon, but also, for example, by the interval equation anomaly detected by W. Lodwick, presented several times in his publications (e.g., Lodwick and Thipwiwatpotjana, 2017).

There are many examples in which the types of IA known today sometimes provide solutions to interval equations and sometimes they do not (Kaczorek and Ruszewski, 2022). Lodwick and Dubois (2015) present an example of a simple equation $[1, 2]x = [4, 6]$ for which the method presented by them gives a tolerance solution being an improper interval of $[4, 3]$, which is interpreted as an empty solution set, because it is not physically feasible. However, for other interval equations, this method provides solutions in the form of proper intervals.

The kind of interval arithmetic that differs from 1-dimensional arithmetic types is multi-dimensional IA (MIA) of Piegat, Pluciński and Landowski. MIA will be presented in Section 2. In their recent articles, the authors presented how to determine realistic tolerance control for the basic 1st order ILE (Piegat and Pluciński, 2022a) and for the quadratic interval equation (Piegat and Pluciński, 2022b). These articles explained the basic concepts and the essence of the proposed method, and the current article is its extension to second- and higher-order ILE systems.

2. Introduction to multidimensional interval arithmetic

The concept of a multidimensional approach to IA was developed in 2010–2011 by A. Piegat in connection with the work on irregular fuzzy models. The first publication on MIA appeared in 2012 (Piegat and Landowski, 2012), soon followed by others (Piegat and Landowski, 2013; Piegat and Pluciński, 2015a). On the basis of MIA with the use of α-cuts, a multidimensional Type-1 fuzzy arithmetic (MFA-Type-1) was developed (e.g., Piegat and Pluciński, 2015b). MFA Type-1 was in turn used to develop MFA Type-2 (e.g., Piegat and Dobryakova, 2020). MIA, MFA Type-1, MFA Type-2 was developed mainly by the research team of A. Piegat, M. Pluciński and M. Landowski. By the beginning of 2022, this team had published 47 articles on the above-mentioned types of arithmetic. Multidimensional arithmetic has met with considerable interest in the world. By the beginning of 2022, other foreign scientists had published over 40 papers presenting various theoretical methods and practical applications of this arithmetic. Examples include the works of Alamanda and Boddeti (2021), Mazandarani *et al*. (2018), or Ngo and Wu (2021).

The most important difference between MIA and 1D-IA is the form and meaning of the calculated result. In MIA, the result is multidimensional and its dimensionality depends on the number of variables involved in the operation. However, in all 1D-IA types, the results of all calculations are 1-dimensional intervals. This approach causes a loss of information the more complex the calculations are. It also causes inaccuracies (sometimes significant) in many, but not all, of the final results. In MIA, the intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ are transformed into the RDM-form $a(\gamma_a)$ and $b(\gamma_b)$, (RDM—relative distance measure):

$$\begin{aligned} &[\underline{a}, \overline{a}] \to \underline{a} + \gamma_a(\overline{a} - \underline{a}), \qquad \gamma_a \in [0, 1], \\ &[\underline{b}, \overline{b}] \to \underline{b} + \gamma_b(\overline{b} - \underline{b}), \qquad \gamma_b \in [0, 1]. \end{aligned} \tag{18}$$

The model (18) is an epistemic model of the true value of the variable a and b (there is only one true value). The operation of adding two intervals is

$$\begin{aligned} &a(\gamma_a) + b(\gamma_b) = c(\gamma_a, \gamma_b), \quad \gamma_a, \gamma_b \in [0, 1], \\ &\quad [\underline{a} + \gamma_a(\overline{a} - \underline{a})] + [\underline{b} + \gamma_b(\overline{b} - \underline{b})] \\ &\qquad = (\underline{a} + \underline{b}) + \gamma_a(\overline{a} - \underline{a}) + \gamma_b(\overline{b} - \underline{b}). \end{aligned} \tag{19}$$

MIA takes into account the fact that, since we do not know the true value of a and b, these variables are unknowns. Can we add two unknowns? Yes! They can be added by treating their possible values as possible hypotheses. Each of the hypotheses will be conditional and will describe one of possible states of the addition system, e.g., IF ($a = 2$) AND ($b = 4$) THEN ($c = 6$). Each of the possible hypotheses is a triple described by

$$(a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b) = a(\gamma_a) + b(\gamma_b)). \tag{20}$$

If $a \in [\underline{a}, \overline{a}] = [2, 3]$ and $b \in [\underline{b}, \overline{b}] = [4, 6]$, examples of possible states of the addition system are given by

$$(2.0, 4.0, 6.0),\ (2.0, 4.1, 6.1),\ (2.1, 4.1, 6.2),\ \dots. \tag{21}$$

The set of possible addition states $S_{A+B} = \{a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b)\}$ is 3-dimensional and the result $c(\gamma_a, \gamma_b)$ depends on two RDM variables. The set S_{A+B} is the main and most informative result of intervals adding. Figure 2 illustrates the operation of addition of two intervals: $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$. It should be added here that in the case of adding three intervals, the set S will be visualized by a 3-dimensional cuboid, not a rectangle as in Fig. 2.

The main result of the addition, the set $S_{A+B} = \{(a, b, c)\}$, is the set of all possible states (a, b, c) of the addition system, of which only one state is the true state. There are many states in the set S_{A+B} containing the same result c with different values of a and b. An example covers the states $(2.0, 5.0, 7.0)$, $(2.1, 4.9, 7.0)$, $(2.2, 4.8, 7.0)$, ... etc. This means that the set of possible c-values contains repeating values and therefore it should be named a bag, not the set. The bag BG_C is defined by

$$\begin{aligned} BG_C = \{&c(\gamma_a, \gamma_b) = a(\gamma_a) + b(\gamma_b) \mid \\ &\forall \gamma_a \in [0, 1], \forall \gamma_b \in [0, 1]\}. \end{aligned} \tag{22}$$

Often, for practical calculations, the span SP_{BG_C} of the bag BG_C is necessary:

$$\begin{aligned} SP_{BG_C} = \Big[\min_{\gamma_a, \gamma_b} c(\gamma_a, \gamma_b), \max_{\gamma_a, \gamma_b} c(\gamma_a, \gamma_b)\Big], \\ \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{23}$$

The transformation of the intervals $[\underline{a}, \overline{a}] = [2, 3]$ and $[\underline{b}, \overline{b}] = [4, 6]$ into the RDM form is shown by

$$\begin{aligned} &[2, 3] \to a(\gamma_a) = 2 + \gamma_a, \quad \gamma_a \in [0, 1], \\ &[4, 6] \to b(\gamma_b) = 4 + 2\gamma_b, \quad \gamma_b \in [0, 1], \\ &a(\gamma_a) + b(\gamma_b) = c(\gamma_a, \gamma_b) = 6 + \gamma_a + 2\gamma_b. \end{aligned} \tag{24}$$

Using the formula (23), we can determine the span

$$SP_{BG_C} = [6, 9], \tag{25}$$

which does not contain the repeating values of the result variable c. The meaning of the span is easy to understand from Fig. 2. The span is the secondary, derivative information about the set S_{A+B} of possible states (a, b, c). However, this information is poorer than the full, 3D set S_{A+B}. On the basis of the span, it is impossible to reconstruct the set S_{A+B}. However, on the basis of the set S_{A+B}, it is easy to determine its span.

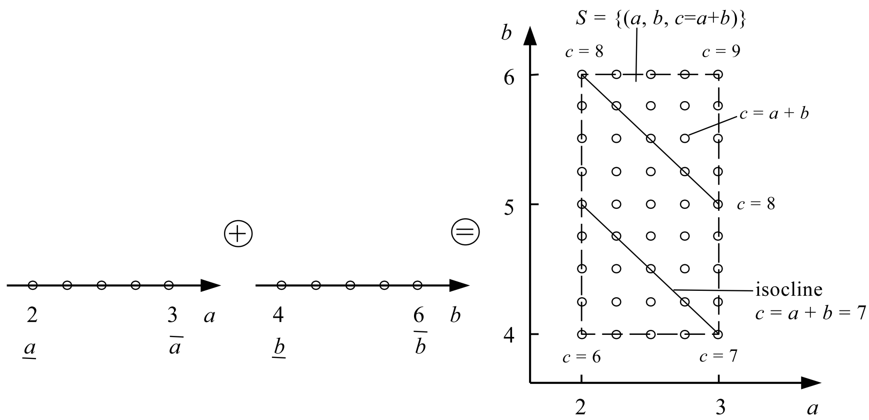

Fig. 2. Visualization of addition of two interval sets $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ and the 3D-result: the set $S_{A+B}\{a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b)\}$ of possible states of the addition system in the projection on 2D-space $A \times B$.

Another kind of simplified, secondary information about the set S_{A+B} of possible states of the addition system is the cardinality density function $\mathit{carddf}(c)$ of the result variable $c(\gamma_a, \gamma_b)$ shown in Fig. 3. In the case of adding two intervals, the distribution $\mathit{carddf}(c)$ can be obtained from the length of the isocline segments $c = a + b =$ const. Two such isoclines are shown in Fig. 2 for $c = a + b = 7$ and $c = 8$. A measure of the cardinality density is the isocline length corresponding to a given value of c. The area under the $\mathit{carddf}(c)$ plot is a measure of the number of all possible states (a, b, c) of the addition system leading to the results $c \in [6, 9]$ (only one of these states will really exist). The plot in Fig. 3 shows that the number of possible system states leading to $c \in [7, 8]$ is two times greater than that of states leading to $c \in [6, 7]$. In other words, the conclusion $c \in [7, 8]$ is consistent with (is supported by) the two times greater number of possible states (a, b, c) that may occur than the conclusion $c \in [6, 7]$. In yet another way the robustness of the conclusion $c \in [7, 8]$ to the uncertainty of the system states is two times higher than the robustness of the conclusion $c \in [6, 7]$. Thus, the $\mathit{carddf}(c)$ distribution in Fig. 3 can be interpreted in three ways: as the cardinality density distribution, as the density distribution of the conclusion adjustment $c = c^*$ to the set of possible states of the system, and as the density distribution of the robustness of the conclusion $c = c^*$ on the set of uncertain states (a, b, c) of the addition system, where c^* is a number selected from the range $[\underline{c}, \overline{c}]$, here equal to the interval $[6, 9]$.

The distributions $\mathit{carddf}(c)$ of the results of addition and subtraction, as linear operations, are created on the basis of straight line segments. On the other hand, the distributions of the results of multiplication and division, as nonlinear operations, are created on the basis of curved segments.

The third instance of simplified, secondary

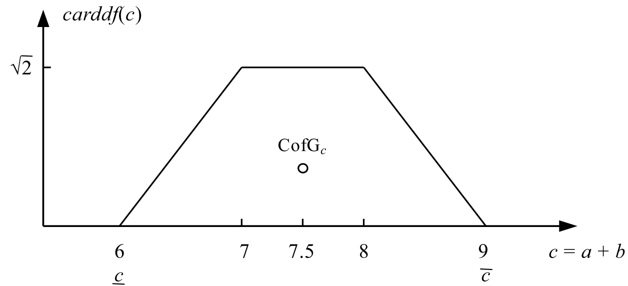

Fig. 3. Distribution $\mathit{carddf}(c)$: the cardinality density function of possible result values $c = a+b$ in the interval addition $[2, 3] + [4, 6]$.

information about the set S_{A+B} of possible states $(a, b, c = a + b)$ of the addition is the center of gravity CofG_C giving the expected value of the result variable c:

$$\mathit{CofG}_C = \frac{\int_{\underline{c}}^{\overline{c}} c \cdot \mathit{carddf}(c)\,\mathrm{d}c}{\int_{\underline{c}}^{\overline{c}} \mathit{carddf}(c)\,\mathrm{d}c} = 7.5\,. \tag{26}$$

The description of the results of addition of the intervals $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}]$ shows that in the case of MIA there is no single calculation result as there is in one-dimensional IA. Here we have one main result (3D), i.e., the set S_{A+B} of possible states of the addition system, and three secondary results derived from the main result: the span SP_{BG_C}, the distribution $\mathit{carddf}(c)$, and the center of gravity CofG_C. 1D IA suggests that the main and only result is the span SP_{BG_C}, which is actually only a secondary result and which is informatively quite poor. Further arithmetic operations are presented below.

Subtraction of intervals $[\underline{a}, \overline{a}] - [\underline{b}, \overline{b}]$: The result variable

$c(\gamma_a, \gamma_b)$ is given by

$$\begin{aligned} c(\gamma_a, \gamma_b) &= a(\gamma_a) - b(\gamma_b) \\ &= [\underline{a} + \gamma_a(\overline{a} - \underline{a})] - [\underline{b} + \gamma_b(\overline{b} - \underline{b})] \\ &= (\underline{a} - \underline{b}) + \gamma_a(\overline{a} - \underline{a}) - \gamma_b(\overline{b} - \underline{b}), \\ &\quad \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{27}$$

The resulting set S_{A-B} of states of the subtraction system is given by

$$\begin{aligned} S_{A-B} = \{(a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b) = a(\gamma_a) - b(\gamma_b)) \mid \\ \forall \gamma_a \in [0, 1],\ \forall \gamma_b \in [0, 1]\}. \end{aligned} \tag{28}$$

Multiplication of intervals $[\underline{a}, \overline{a}] \cdot [\underline{b}, \overline{b}]$: The result variable $c(\gamma_a, \gamma_b)$ is given by

$$c(\gamma_a, \gamma_b) = a(\gamma_a) b(\gamma_b), \quad \gamma_a, \gamma_b \in [0, 1]. \tag{29}$$

The resulting set S_{AB} of possible states of the mutiplication is given by

$$\begin{aligned} S_{AB} = \{(a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b) = a(\gamma_a) b(\gamma_b)) \mid \\ \forall \gamma_a \in [0, 1],\ \forall \gamma_b \in [0, 1]\}. \end{aligned} \tag{30}$$

Dividing proper intervals $[\underline{a}, \overline{a}] / [\underline{b}, \overline{b}]$: The result variable $c(\gamma_a, \gamma_b)$ is given by

$$c(\gamma_a, \gamma_b) = a(\gamma_a) / b(\gamma_b), \quad 0 \notin b(\gamma_b),\ \gamma_a, \gamma_b \in [0, 1]. \tag{31}$$

The resulting set $S_{A/B}$ of possible states of the dividing system is given by

$$\begin{aligned} S_{A/B} = \{(a(\gamma_a), b(\gamma_b), c(\gamma_a, \gamma_b) = a(\gamma_a) / b(\gamma_b)) \mid \\ \forall \gamma_a \in [0, 1],\ \forall \gamma_b \in [0, 1]\}. \end{aligned} \tag{32}$$

If $[\underline{b}, \overline{b}]$ contains zero, which can be written as $[\underline{b}, 0, \overline{b}]$, where $\underline{b} < 0$, $\overline{b} > 0$, then the division operation can be performed approximately using the decomposition of the interval $[\underline{b}, 0, \overline{b}]$ into two component intervals excluding zero:

$$[\underline{b}, 0, \overline{b}] = [\underline{b}, -\Delta] \cup [\Delta, \overline{b}], \tag{33}$$

where Δ is a very small number, such as $\Delta = 0.001$.

The entire division operation can also be decomposed into a union of 2-component division operations:

$$[\underline{a}, \overline{a}] / [\underline{b}, 0, \overline{b}] = [\underline{a}, \overline{a}] / [\underline{b}, -\Delta] \cup [\underline{a}, \overline{a}] / [\Delta, \overline{b}]. \tag{34}$$

In summary, the most important benefits of using MIA are as follows:

- Almost all laws of the arithmetic of crisp numbers hold for RDM arithmetic.
- Complicated problems can be solved, thanks to possibility of an equation's transformation.
- MIA provides complete, multidimensional problem solutions from which various simplified representations can be derived.

In Section 3, MIA will be applied to solve interval equations with one unknown.

3. Determining the robustness of solutions of interval equations of order 1

First, the solution method will be presented for the case of one control variable, because only then this method can be visualized and the basic concepts explained understandably. In the case of two control variables, the problem becomes high-dimensional and its visualization is much complicated.

Solving an ILE of order 1 can be formulated as follows. Given is a static, multiplicative system with inputs a and x that performs the operations $ax = b$. We only have an approximate knowledge about the input a: $a \in [\underline{a}, \overline{a}]$. We also know the requirement for the output tolerance corridor b: $b \in [\underline{b}, \overline{b}]$. Specify a value of the control input x (or the set $X = [\underline{x}, \overline{x}]$ of these values) that allows 'hiting' the output b, ($ax \in [\underline{b}, \overline{b}]$). Although the variables a and x are physical inputs of the system under consideration and b is its output, in terms of information a and b are information inputs and the control x becomes the information output. The algorithm for solving this problem will be presented using the example of the system $[2, 3]x = [3, 6]$. This example was studied by Lodwick and Dubois (2015). Both SIA and CIA detected only a partial range of solutions here.

Example 1. (*Solving steps*)

Step 1: Formulate the interval sets $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ in terms of MIA:

$$\begin{aligned} [\underline{a}, \overline{a}] = [2, 3] &\rightarrow a(\gamma_a) = \underline{a} + \gamma_a(\overline{a} - \underline{a}) = 2 + \gamma_a, \\ [\underline{b}, \overline{b}] = [3, 6] &\rightarrow b(\gamma_b) = \underline{b} + \gamma_b(\overline{b} - \underline{b}) = 3 + 3\gamma_b, \\ &\quad \gamma_a, \gamma_b \in [0, 1]. \end{aligned} \tag{35}$$

Step 2: Determine the bag model BG_X:

$$\begin{aligned} ax = b \rightarrow x(\gamma_a, \gamma_b) = \frac{b(\gamma_b)}{a(\gamma_a)} = \frac{3 + 3\gamma_b}{2 + \gamma_a}, \\ BG_X = \Big\{ x(\gamma_a, \gamma_b) = \frac{3 + 3\gamma_b}{2 + \gamma_a} \mid \\ \forall \gamma_a \in [0, 1], \forall \gamma_b \in [0, 1] \Big\}. \end{aligned} \tag{36}$$

Step 3: Determine the bag span $SP_{BG_X} = X_{poss} = [\underline{x}, \overline{x}]$:

$$SP_{BG_X} = \Big[\min_{\gamma_a,\gamma_b} \frac{3+3\gamma_b}{2+\gamma_a}, \max_{\gamma_a,\gamma_b} \frac{3+3\gamma_b}{2+\gamma_a}\Big] \tag{37}$$
$$= [1,3] = [\underline{x}, \overline{x}] = X_{poss}\,.$$

A minimum of $x(\gamma_a, \gamma_b)$ was obtained for $\gamma_a = 1$, $\gamma_b = 0$, and a maximum of $x(\gamma_a, \gamma_b)$ for $\gamma_a = 0$, $\gamma_b = 1$.

Step 4: Determine the degrees of robustness $r(x)$ for x-values contained in X_{poss}.

Figure 4 illustrates the meaning of the bag BG_X of possible system states $(a, b, x = b/a)$ satisfying the control objective as well as the meaning of the span $SP_{BG_X} = X_{poss} = [\underline{x}, \overline{x}]$.

Figure 4 shows that the control range $X_{poss} = [1, 3]$ enabling (but not always ensuring) a hit in the tolerance corridor TC = $[3, 6]$ includes the sub-range $X_{FR} = [1.5, 2]$ of full control robustness x to all possible DUG values $a \in [2, 3]$. X_{poss} also includes 2 sub-ranges of partial robustness. The sub-range $X_{PR1} = [1, 1.5]$ is robust to large values of the variable a, but not robust to small values of a. On the other hand, the sub-range $X_{PR2} = [2, 3]$ has robustness to small values of disturbance a, but does not have robustness to large disturbances a. In Fig. 4 it is possible to distinguish some characteristic values of x defining the points of intersection of the TC borders $ax = 3$ and $ax = 6$ with the lower and upper disturbance values $a = 2$ and $a = 3$. These are the points $x = 1$, $x = 1.5$, $x = 2$, $x = 6$.

The robustness of the control (decision) value x can be interpreted as a fraction, $r(x) \in [0, 1]$, telling to how large a part of the a-disturbance range the selected x value is robust.

In the range $X_{FR} = [1.5, 2]$, the value $r(x) = 1$. In the range $X_{PR1} = [1, 1.5]$, the robustness is given by

$$r(x) = \frac{Mcard(a_{hit}(x))}{Mcard(a)} = \frac{\overline{a} - a(x)}{\overline{a} - \underline{a}} = \frac{\overline{a} - (\underline{b}/x)}{\overline{a} - \underline{a}} \tag{38}$$

and in the range $X_{PR2} = [2, 3]$ by

$$r(x) = \frac{a(x) - \underline{a}}{\overline{a} - \underline{a}} = \frac{(\overline{b}/x) - \underline{a}}{\overline{a} - \underline{a}}. \tag{39}$$

Figure 5 shows the distribution of $r(x)$.

Step 5: Determine the optimal tolerance control value x_{opt} based on the criterion of maximal robustness.

Determining x_{opt} is easy in this problem. One of the x-values within the range $[1.5, 2]$ can be selected. It should be noted that the optimal range of x is determined by two internal characteristic values: $x = 1.5$ and $x = 2$. However, in most problems the choice will not be that easy because the range of full robustness will not exist at all. This is due to the large number of DUGs found in practical problems. The following example will show how increasing the uncertainty of the variable $a \in [\underline{a}, \overline{a}]$ affects the achieved control robustness. ♦

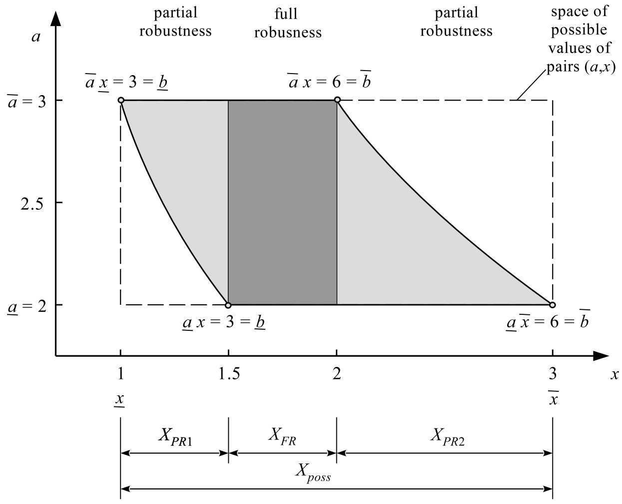

Fig. 4. Visualization of the solution of the task of determining tolerant control of the system defined by the equation $[2, 3]x = [3, 6]$. The tolerance corridor is limited by two curves: $ax = 3$ and $ax = 6$. FR: full robustness, PR: partial robustness.

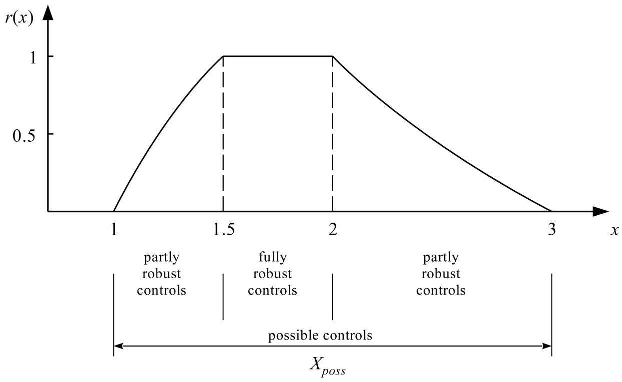

Fig. 5. Distribution of the robustness degree $r(x)$ of particular control values $x \in X_{poss}$ in the tolerant control problem $[2, 3]x = [3, 6]$.

Example 2. The system $[2, 5]x = [3, 6]$ will be considered. Here the width of the disturbance uncertainty $[\underline{a}, \overline{a}] = [2, 5]$ is greater than in the system (35): $[2, 3]x = [3, 6]$. This example has not been previously explored in publications of other authors. Figure 6 shows the solution of the problem in the $A \times X$ space.

The robustness distribution $r(x)$ from Fig. 7 shows that there is no control value x that guarantees full robustness $r(x) = 1$ for the disturbance $[\underline{a}, \overline{a}] = [2, 5]$. Thus, in this task the possibilities of implementing of TC worsened in relation to the task (35) with $[\underline{a}, \overline{a}] =$

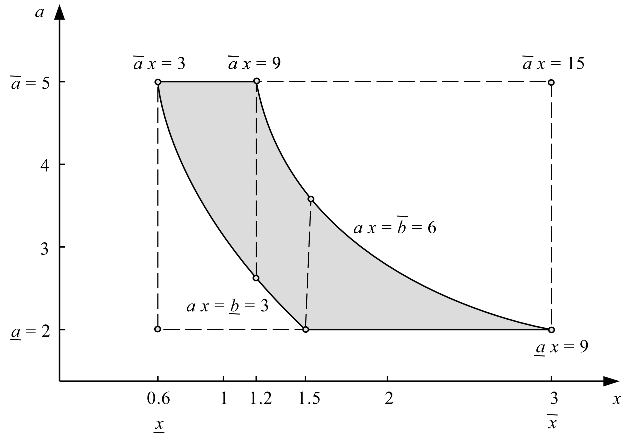

Fig. 6. Visualization of the solution to the problem $[2, 5]x = [3, 6]$ with the uncertainty $[a] = [2, 5]$ increased in relation to the problem (35).

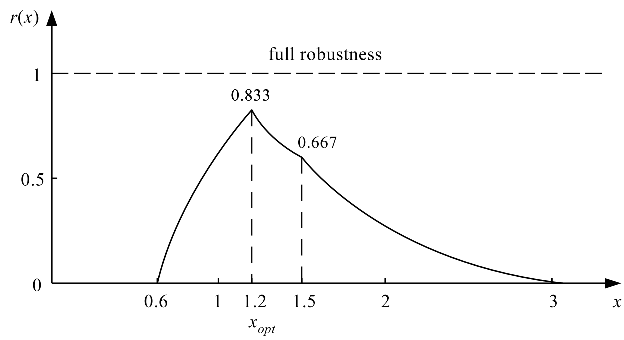

Fig. 7. Robustness distribution of possible control values x in the problem $[2, 5]x = [3, 6]$.

$[2, 3]$. Nevertheless, there is a control value $x_{opt} = 1.2$ which makes it possible to obtain high robustness $r(x_{opt}) = 0.833$. The value of $x_{opt} = 1.2$ is an internal characteristic value of the variable x. ♦

Section 4 presents the problem of determining the robustness of solutions for a system with two control variables (x_1, x_2).

4. Determining the robustness of solutions of interval equation systems

4.1. Example 3. Let us consider the well-known system of equations presented by Barth and Nuding (1974) and later analyzed many times in various publications (Gay, 1982; Lodwick and Dubois, 2015). The methods presented in these articles were unable to calculate the robustness of possible solutions (x_1, x_2) lying outside the area of full robustness (Fig. 11—dark gray area):

$$\begin{aligned} [2,4]\,x_1 + [-2,1]\,x_2 &= [-2,2], \\ [-1,2]\,x_1 + [2,4]\,x_2 &= [-2,2]. \end{aligned} \tag{40}$$

The set of possible solutions of the system (united solutions set) is shown in Fig. 8.

However, to better explain the methodology for determining the robustness of the solution, an initially simplified variant of the system (40) will be considered:

$$\begin{aligned} [2,4]\,x_1 - x_2 &= [-2,2], \\ x_1 + 3x_2 &= [-2,2], \end{aligned} \tag{41}$$

in which there is only one uncertain coefficient on the left-hand side of the system. The coefficients can be written in the form

$$\begin{aligned} a_{11} &= 2 + 2\gamma_{a_{11}}, & b_1 &= -2 + 4\gamma_{b_1}, \\ a_{12} &= -1, & b_2 &= -2 + 4\gamma_{b_2}, \\ a_{21} &= 1, \\ a_{22} &= 3. \end{aligned}$$

The values of the main determinant of the system of equations

$$\Delta = 3(2 + 2\gamma_{a_{11}}) + 1 = 7 + 6\gamma_{a_{11}} \tag{42}$$

belong to the interval $[7, 13]$. The solution of the system (41) is

$$\begin{aligned} x_1 &= \frac{(-2+4\gamma_{b_1})\cdot 3 - (-2+4\gamma_{b_2})\cdot(-1)}{\Delta} \\ &= \frac{-8 + 12\gamma_{b_1} + 4\gamma_{b_2}}{7 + 6\gamma_{a_{11}}}, \\ x_2 &= \frac{(-2+4\gamma_{b_2})\cdot(2+2\gamma_{a_{11}}) - (-2+4\gamma_{b_1})\cdot 1}{\Delta} \\ &= \frac{-2 - 4\gamma_{a_{11}} - 4\gamma_{b_1} + 8\gamma_{b_2} + 8\gamma_{a_{11}}\gamma_{b_2}}{7 + 6\gamma_{a_{11}}}. \end{aligned} \tag{43}$$

The set of possible solutions (united solutions set) is shown in Fig. 9 and additionally also plotted in Fig. 8.

Table 1 gives the values of the 'characteristic solutions' calculated for the boundary values of the coefficients a_{11}, b_1, b_2 for which $\gamma_{a_{11}}, \gamma_{b_1}, \gamma_{b_2} \in \{0, 1\}$. These points are also plotted on the graph in Fig. 9.

Robust solutions (tolerable solutions in the Shary sense) are defined by Eqn. (5). In other words, it is a set of solutions characterized by the property that, for every solution point (x_1, x_2), for every possible matrix A, there exists a vector b such that the equation is satisfied. To determine such a set, it should be checked for each possible solution that for all possible combinations of coefficients of the matrix $\mathbf{A}$, the values on the left-hand sides of the system of equations belong to the intervals on the right-hand sides.

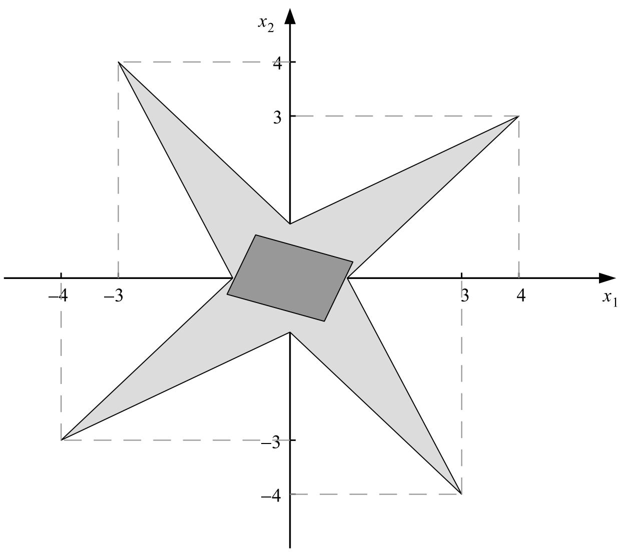

Fig. 8. Set of possible solutions of the system (40) (light gray area) and of the system (41) (dark gray area).

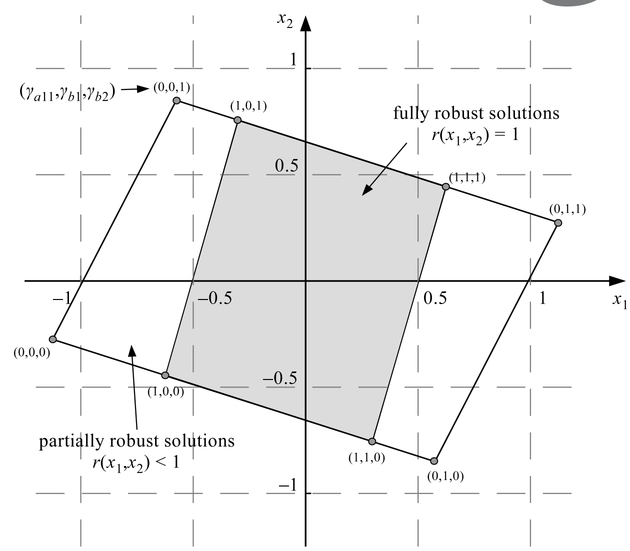

Fig. 9. Set of possible solutions of the system (41).

Table 1. Values of the 'characteristic solutions' of the system (41).

$\gamma_{a_{11}}$	γ_{b_1}	γ_{b_2}	x_1	x_2
0	0	0	$-8/7$	$-2/7$
0	0	1	$-4/7$	$6/7$
0	1	0	$4/7$	$-6/7$
0	1	1	$8/7$	$2/7$
1	0	0	$-8/13$	$-6/13$
1	0	1	$-4/13$	$10/13$
1	1	0	$4/13$	$-10/13$
1	1	1	$8/13$	$6/13$

In the example under consideration, we have only one uncertain coefficient on the left-hand side of the equations:

$$a_{11} = 2 + 2\gamma_{a_{11}}.$$

For each point of the possible solution (x_1, x_2), we can determine an interval $\mathbf{a}_{11r}$ of coefficient $\mathbf{a}_{11}$ values which guarantees that the value of the left-hand side of the first equation is included in the interval $\mathbf{b}_1$.

From the first equation, we can calculate

$$\mathbf{a}_{11} = \frac{x_2 - \mathbf{b}_1}{x_1}.$$

Considering the boundary values $\underline{b}_1 = -2$ and $\overline{b}_1 = 2$, we can determine

$$\mathbf{a}_{11r} = \begin{cases} \left[\frac{x_2-2}{x_1}, \frac{x_2+2}{x_1}\right] & \text{for} \quad x_1 > 0, \\ \left[\frac{x_2+2}{x_1}, \frac{x_2-2}{x_1}\right] & \text{for} \quad x_1 < 0, \\ \mathbb{R} & \text{for} \quad x_1 = 0. \end{cases} \tag{44}$$

If the intersection of intervals $\mathbf{a}_{11r}$ and $\mathbf{a}_{11}$ is equal to $\mathbf{a}_{11}$,

$$\mathbf{a}_{11r} \cap \mathbf{a}_{11} = \mathbf{a}_{11}, \tag{45}$$

then the tested point (x_1, x_2) is a robust solution. The set of fully robust solutions of the system (41) is marked in gray in Fig. 9.

If the condition (45) is not satisfied, we can speak about lack of full robustness and existence of only partial robustness. The intersection of the intervals $\mathbf{a}_{11r}$ and $\mathbf{a}_{11}$ can be denoted as $\mathbf{a}_{11hit}(x)$. If

- $Mcard(\mathbf{a}_{11hit}(x))$—the width of the intersection of the interval $\mathbf{a}_{11r}$ and $\mathbf{a}_{11}$,
- $Mcard(\mathbf{a}_{11})$— the width of the interval a_{11},

then the robustness of the solution can be calculated as

$$r(x_1, x_2) = \frac{Mcard(\mathbf{a}_{11hit}(x))}{Mcard(\mathbf{a}_{11})}. \tag{46}$$

For solutions that belong to the tolerable solutions set we get $r(x_1, x_2) = 1$.

For the system considered, it is sufficient to analyze only the first equation when determining the robustness. The second equation for any point that is a possible solution will always be satisfied, since all the coefficients on the left-hand side are certain.

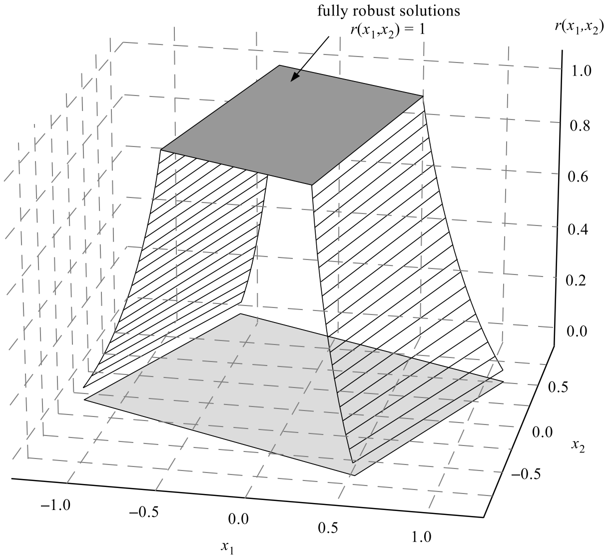

Fig. 10. Robustness $r(x_1, x_2)$ of possible solutions of the system (41).

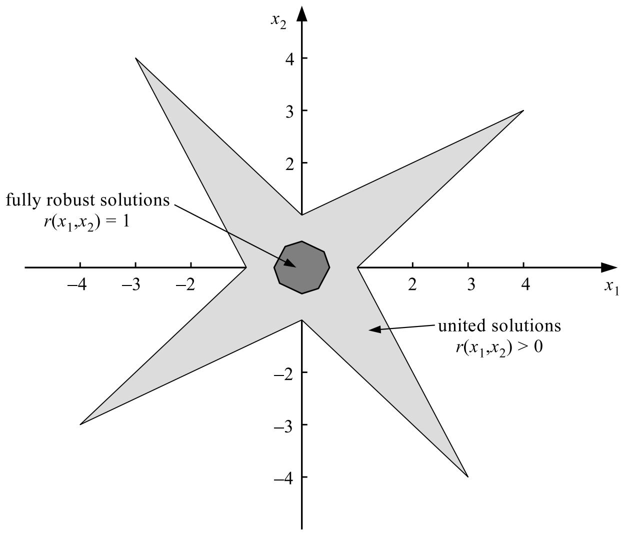

Fig. 11. Solutions of the system (40): the set of possible (united) solutions (light gray area), the set of fully robust (tolerable) solutions (dark gray area).

In practice, robustness calculations can also be performed numerically. For each point of possible solutions (x_1, x_2), we can greedily examine the values of the interval $a_{11} = [2, 4]$ with a sufficiently small step, testing whether, when substituted into the system of equations, the values on the left-hand side belong to the intervals on the right-hand side. The number of the values a_{11} that satisfy the system divided by the number of all tested values will be an approximation of the robustness value $r(x_1, x_2)$ for the point (x_1, x_2). Of course, the higher the sampling density of the interval we take, the better the determined value will be as an approximation of robustness. Figure 10 shows the robustness $r(x_1, x_2)$ graph for possible solutions of the system.

Let us now return to the initial system of equations (40). The coefficients can be written in the form

$$
\begin{aligned}
a_{11} &= 2 + 2\gamma_{a_{11}}, & b_1 &= -2 + 4\gamma_{b_1},\\
a_{12} &= -2 + 3\gamma_{a_{12}}, & b_2 &= -2 + 4\gamma_{b_2},\\
a_{21} &= -1 + 3\gamma_{a_{21}}, & &\\
a_{22} &= 2 + 2\gamma_{a_{22}}. & &
\end{aligned}
$$

The values of the main determinant of the system of equations,

$$
\begin{aligned}
\Delta = {} & (2 + 2\gamma_{a_{11}})(2 + 2\gamma_{a_{22}})\\
& - (-2 + 3\gamma_{a_{12}})(-1 + 3\gamma_{a_{21}}),
\end{aligned}
\tag{47}
$$

belong to the interval $[2, 20]$. The solution of the system (40) is

$$
\begin{aligned}
x_1 = \frac{1}{\Delta}\Big[& (-2 + 4\gamma_{b_1})(2 + 2\gamma_{a_{22}})\\
& - (-2 + 4\gamma_{b_2})(-2 + 3\gamma_{a_{12}})\Big],
\end{aligned}
$$

$$
\begin{aligned}
x_2 = \frac{1}{\Delta}\Big[& (-2 + 4\gamma_{b_2})(2 + 2\gamma_{a_{11}})\\
& - (-2 + 4\gamma_{b_1})(-1 + 3\gamma_{a_{21}})\Big].
\end{aligned}
\tag{48}
$$

The set of possible solutions is shown in Fig. 11.

The calculation of the robustness of the solutions should be carried out for each point (x_1, x_2) that is a possible solution to the system of equations.

Theorem 1. *We can determine the robustness for each equation separately ($r_1(x_1, x_2)$ for the first equation and $r_2(x_1, x_2)$ for the second equation), and calculate the robustness of the whole system as*

$$
r(x_1, x_2) = r_1(x_1, x_2) \cdot r_2(x_1, x_2)\,. \tag{49}
$$

For a larger number of equations with more unknowns, the calculations can similarly be performed for each equation separately, and the total robustness of the system can be calculated as the product of the robustness terms determined for each equation.

Proof. The proof will be provided for a system of linear equations of 2nd order, but it is easy to adapt it for systems of higher orders. Given is an ILS defined by

$$
\begin{aligned}
& a_{11}x_1 + a_{12}x_2 = b_1,\\
& a_{21}x_1 + a_{22}x_2 = b_2,\\
& \quad a_{11} \in [\underline{a}_{11}, \overline{a}_{11}], \quad a_{12} \in [\underline{a}_{12}, \overline{a}_{12}], \quad b_1 \in [\underline{b}_1, \overline{b}_1],\\
& \quad a_{21} \in [\underline{a}_{21}, \overline{a}_{21}], \quad a_{22} \in [\underline{a}_{22}, \overline{a}_{22}], \quad b_2 \in [\underline{b}_2, \overline{b}_2].
\end{aligned}
\tag{50}
$$

The values of the four coefficients $(a_{11}, a_{12}, a_{21}, a_{22})$ are independent of us. The intervals $[\underline{b}_1, \overline{b}_1]$ and $[\underline{b}_2, \overline{b}_2]$, on the other hand, are determined

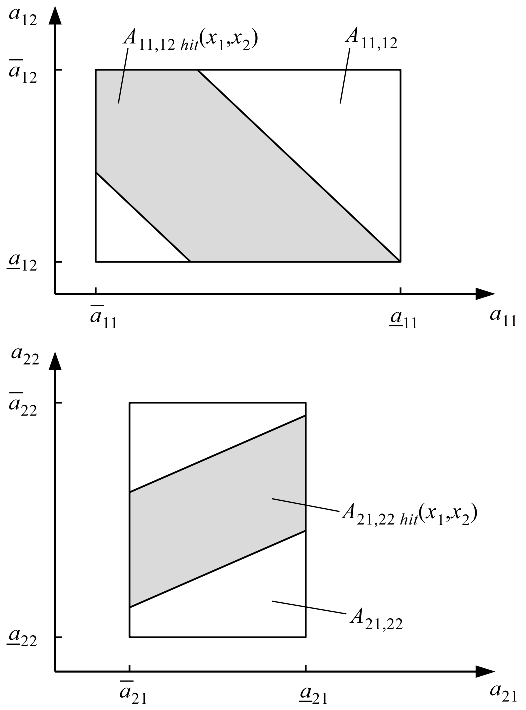

Fig. 12. Illustration of sets of possible values of tuples $A_{11,12}$ and $A_{21,22}$ as well as sets of hits in tolerances $[\underline{b}_1, \overline{b}_1]$ and $[\underline{b}_2, \overline{b}_2]$ for a single possible control pair (x_1, x_2).

by the system expert and are the tolerances into which the values of the left-hand sides of the equations in (50) should hit. Let us define the sets $A_{11,12}$, $A_{21,22}$ and $A_{11,12,21,22}$:

$$\begin{aligned} A_{11,12} = \{&(a_{11}, a_{12}) \mid \\ &(a_{11} \in [\underline{a}_{11}, \overline{a}_{11}]), (a_{12} \in [\underline{a}_{12}, \overline{a}_{12}])\}, \\ A_{21,22} = \{&(a_{21}, a_{22}) \mid \\ &(a_{21} \in [\underline{a}_{21}, \overline{a}_{21}]), (a_{22} \in [\underline{a}_{22}, \overline{a}_{22}])\}, \end{aligned}$$

$$\begin{aligned} &A_{11,12,21,22} \\ &= \{(a_{11}, a_{12}, a_{21}, a_{22}) \mid \\ &\quad (a_{11} \in [\underline{a}_{11}, \overline{a}_{11}]),\ (a_{12} \in [\underline{a}_{12}, \overline{a}_{12}]), \\ &\quad (a_{21} \in [\underline{a}_{21}, \overline{a}_{21}]), (a_{22} \in [\underline{a}_{22}, \overline{a}_{22}])\}. \end{aligned} \tag{51}$$

It can be seen that $A_{11,12,21,22} = A_{11,12} \times A_{21,22}$. The above are the sets of all possible tuples (a_{11}, a_{12}), (a_{21}, a_{22}) and quadruples $(a_{11}, a_{12}, a_{21}, a_{22})$ that can exist in the ILS (50). The sets of $A_{11,12}$ and $A_{21,22}$ are visualized in Fig. 12.

Let us now define the sets $A_{11,12hit}$, $A_{21,22hit}$ and $A_{11,12,21,22hit}$. These depend on the chosen pair of controls (x_1, x_2):

$$\begin{aligned} &A_{11,12hit}(x_1, x_2) \\ &= \{(a_{11}, a_{12}) \mid (a_{11}x_1 + a_{12}x_2 = b_1 \in [\underline{b}_1, \overline{b}_1]), \\ &\quad ((x_1, x_2) \in USS)\} \\ &A_{21,22hit}(x_1, x_2) \\ &= \{(a_{21}, a_{22}) \mid (a_{21}x_1 + a_{22}x_2 = b_2 \in [\underline{b}_2, \overline{b}_2]), \\ &\quad ((x_1, x_2) \in USS)\} \\ &A_{11,12,21,22hit}(x_1, x_2) \\ &= \{(a_{11}, a_{12}, a_{21}, a_{22})) \mid \\ &\quad (a_{11}x_1 + a_{12}x_2 = b_1 \in [\underline{b}_1, \overline{b}_1]), \\ &\quad (a_{21}x_1 + a_{22}x_2 = b_2 \in [\underline{b}_2, \overline{b}_2]), \\ &\quad ((x_1, x_2) \in USS)\}. \end{aligned} \tag{52}$$

$A_{11,12hit}(x_1, x_2)$ is the set of tuples (a_{11}, a_{12}) that are matched to the selected pair (x_1, x_2) in the sense that together they give the solution of the first equation $a_{11}x_1 + a_{12}x_2 = b_1$, $b_1 \in [\underline{b}_1, \overline{b}_1]$, which means that the value b_1 dependent on them hits the required tolerance interval $[\underline{b}_1, \overline{b}_1]$. The set $A_{21,22hit}(x_1, x_2)$ has a similar meaning with respect to the second equation. It is easy to figure out that $A_{11,12,21,22hit}(x_1, x_2)$ is the set of quadruples $(a_{11}, a_{12}, a_{21}, a_{22})$ such that, when combined with a pair of controls (x_1, x_2), yield the fulfillment of both the equations in (50), that is, hit both tolerances $[\underline{b}_1, \overline{b}_1]$ and $[\underline{b}_2, \overline{b}_2]$ simultaneously. Such quadruples $(a_{11}, a_{12}, a_{21}, a_{22})$ must therefore be composed of tuples $(a_{11}, a_{12}) \in A_{11,12hit}(x_1, x_2)$ and $(a_{21}, a_{22}) \in A_{21,22hit}(x_1, x_2)$. It follows that

$$\begin{aligned} &A_{11,12,21,22hit}(x_1, x_2) \\ &= (A_{11,12hit}(x_1, x_2)) \times (A_{21,22hit}(x_1, x_2)). \end{aligned} \tag{53}$$

A similar formula can be written for the sets $A_{11,12}$, $A_{21,22}$ and $A_{11,12,21,22}$:

$$A_{11,12,21,22} = (A_{11,12}) \times (A_{21,22}). \tag{54}$$

If all sets of tuples and quadruples occurring in the problem are continuous, then the number of elements included in them is infinitely large. In such a situation, a measure *Mcard* of their quantity can be used to perform calculations. The formulas (55) give measures of quantity (cardinality) for particular sets:

$$\begin{aligned} &Mcard(A_{11,12}) = (\overline{a}_{11} - \underline{a}_{11})(\overline{a}_{12} - \underline{a}_{12}), \\ &Mcard(A_{21,22}) = (\overline{a}_{21} - \underline{a}_{21})(\overline{a}_{22} - \underline{a}_{22}), \\ &Mcard(A_{11,12,21,22}) \\ &\quad = (\overline{a}_{11} - \underline{a}_{11})(\overline{a}_{12} - \underline{a}_{12})(\overline{a}_{21} - \underline{a}_{21})(\overline{a}_{22} - \underline{a}_{22}) \\ &\quad = (Mcard\ A_{11,12}) \cdot (Mcard\ A_{21,22}), \\ &Mcard(A_{11,12hit}(x_1, x_2)) = Area\ A_{11,12hit}(x_1, x_2), \\ &Mcard(A_{21,22hit}(x_1, x_2)) = Area\ A_{21,22hit}(x_1, x_2), \\ &Mcard(A_{11,12,21,22hit}(x_1, x_2)) \\ &\quad = HV A_{11,12,21,22hit}(x_1, x_2) \\ &\quad = (Area\ A_{11,12hit}(x_1, x_2)) \\ &\qquad \cdot (Area\ A_{21,22hit}(x_1, x_2)). \end{aligned} \tag{55}$$

The HV notation is the abbreviation for the hyper-volume, meaning the volume of a geometric solid in the 4D space. Such a solid is a relation of geometric figures existing in the 2D space $(A_{11,12hit}(x_1,x_2)) \times (A_{21,22hit}(x_1,x_2))$, see Fig. 12. Geometric figures defined by the sets $A_{11,12hit}(x_1,x_2)$ and $A_{21,22hit}(x_1,x_2)$ are always convex, hence HV of a 4D solid can be calculated as the product $(Area\, A_{11,12hit}(x_1,x_2)) \cdot (Area\, A_{21,22hit}(x_1,x_2))$.

The robustness $r(x_1,x_2)$ of a pair of controls (x_1,x_2) to the uncertainty of coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ can be determined by

$$\begin{aligned} r(x_1,x_2) &= \frac{Mcard(A_{11,12,21,22hit}(x_1,x_2))}{Mcard(A_{11,12,21,22})} \\ &= \frac{Mcard(A_{11,12hit}(x_1,x_2))}{Mcard(A_{11,12})} \cdot \frac{Mcard(A_{21,22hit}(x_1,x_2))}{Mcard(A_{21,22})} \\ &= r_1(x_1,x_2) \cdot r_2(x_1,x_2). \end{aligned} \tag{56}$$

The formula (56) means that the measure of robustness $r(x_1,x_2)$ for a pair of controls (x_1,x_2) of a system described by the two equations in (50) can be calculated from the measures of robustness $r_1(x_1,x_2)$ and $r_2(x_1,x_2)$ calculated separately for each equation of the system. This significantly reduces the computational complexity of the problem.

Extension of the formulas (56) to an ILS of higher orders should not be a problem. Only the number of equations for which $r_i(x_1,x_2,\ldots,x_n)$, $i=1,\ldots,n$, must be separately determined increases, and then their product is calculated, which is the total robustness of the analyzed system.

■

To calculate the robustness of the first equation, we must first determine the span of the resulting interval on the left-hand side of the equation and check whether its intersection with the right-hand side is not empty.

1. If the intersection is empty, the robustness of the equation one $r_1(x_1,x_2)=0$ (and thus the robustness of the whole system $r(x_1,x_2)=0$). The calculations can be stopped here.

2. If the interval on the left is completely included in the interval on the right, then the robustness of the first equation $r_1(x_1,x_2)=1$.

3. If the interval on the left is partially included in the interval on the right, we must perform the following steps:

   - Create an empty set of points S_1.
   - Determine the values of the expression on the left $L_1(x_1,x_2) = a_{11}x_1 + a_{12}x_2$ for all combinations of boundary values of a_{11} and a_{12}, that is, for the points
   $$(\underline{a_{11}}, \underline{a_{12}}),\ (\underline{a_{11}}, \overline{a_{12}}),\ (\overline{a_{11}}, \underline{a_{12}}),\ (\overline{a_{11}}, \overline{a_{12}}). \tag{57}$$
   - Add to the set of points S_1 those for which the value of the expression $L_1(x_1,x_2)$ belongs to the interval b_1.
   - Find all points of the intersection of lines:
   $$a_{11}x_1 + a_{12}x_2 = \underline{b_1}, \quad a_{11}x_1 + a_{12}x_2 = \overline{b_1},$$
   with four segments defined by the points (57) and add them to the set S_1.
   - Remove repeating points from the set.
   - If the number of points in the set S_1 is less than three, then the robustness $r_1(x_1,x_2)=0$. Otherwise, calculate the area of the convex figure $AR_{1hit}(x_1,x_2)$ defined by points from the set S_1 (the maximum number of points is six).
   - Calculate the robustness of the first equation as
   $$r_1(x_1,x_2) = \frac{Mcard(A_{11,12hit}(x_1,x_2))}{Mcard(A_{11,12})} = \frac{AR_{1hit}(x_1,x_2)}{AR_1},$$
   where AR_1 is the area of the rectangle described by the points (57).

For the second equation, the robustness $r_2(x_1,x_2)$ can be determined similarly. The robustness of the whole system can be calculated from Eqn. (49).

Let us perform sample calculations for the point $(x_1,x_2)=(-1,1)$.

Figure 13 shows a rectangle defined by intervals of the coefficients $a_{11}=[2,4]$ and $a_{12}=[-2,1]$. The values of the left-hand side of the first equation L_1 at corner points (shown in italics in the figure) are calculated. The intersection points of the line $a_{11}x_1 + a_{12}x_2 = \underline{b_1}$ with sides of the rectangle are also determined. Next, we determine a set consisting of the points that are relevant to us (for which the value L_1 belongs to the b_1 interval). In the set we have three points:

$$S_1 = \{(2,1),(2,0),(3,1)\},$$

which define a triangle. The area of the triangle $AR_{1hit}(x_1,x_2)=0.5$, and the area $AR_1=6$, hence the robustness of the first equation at the point $(x_1,x_2)=(-1,1)$ is

$$r_1(x_1,x_2) = \frac{AR_{1hit}(x_1,x_2)}{AR_1} = \frac{1}{12}.$$

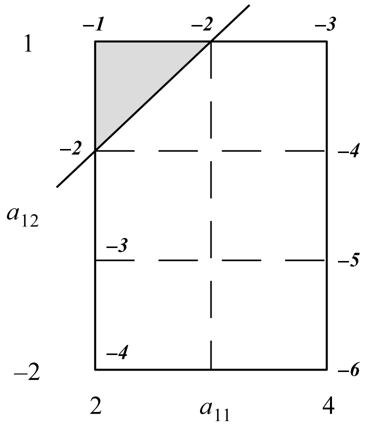

Fig. 13. Illustration for calculations at the point $(x_1, x_2) = (-1, 1)$ for the first equation.

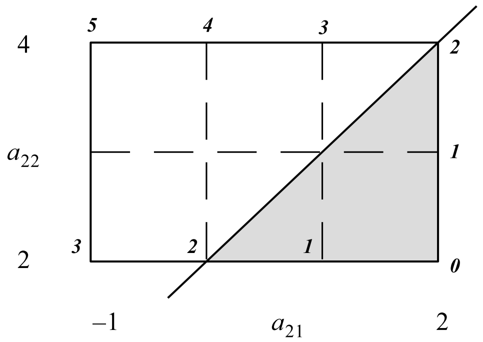

Fig. 14. Illustration for calculations at the point $(x_1, x_2) = (-1, 1)$ for the second equation.

Figure 14 shows a rectangle defined by intervals of the coefficients $a_{21} = [-1, 2]$ and $a_{22} = [2, 4]$. The values of the left-hand side of the second equation L_2 at corner points (shown in italics in the figure) are calculated. The intersection points of the line $a_{21}x_1 + a_{22}x_2 = \overline{b_2}$ with sides of the rectangle are also determined. Next, we determine a set consisting of the points that are relevant to us (for which the value L_2 belongs to the interval b_2). In the set we have three points:

$$S_2 = \{(2, 2), (0, 2), (2, 4)\},$$

which define a triangle. The area of the triangle $AR_{2hit}(x_1, x_2) = 2$, and the area $AR_2 = 6$; hence the robustness of the second equation at the point $(x_1, x_2) = (-1, 1)$ is

$$r_2(x_1, x_2) = \frac{AR_{2hit}(x_1, x_2)}{AR_2} = \frac{2}{6}.$$

We get

$$r(x_1, x_2) = r_1(x_1, x_2) \cdot r_2(x_1, x_2) = \frac{1}{12} \cdot \frac{2}{6} = \frac{1}{36}.$$

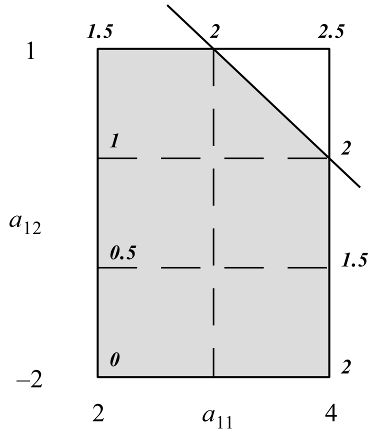

Fig. 15. Illustration for calculations at the point $(x_1, x_2) = (0.5, 0.5)$ for the first equation.

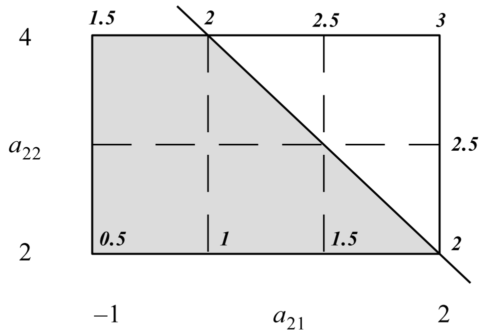

Fig. 16. Illustration for calculations at the point $(x_1, x_2) = (0.5, 0.5)$ for the second equation.

In order to better explain the process of calculations, let us next determine the robustness at the point $(x_1, x_2) = (0.5, 0.5)$.

Figure 15 shows a rectangle defined by intervals of the coefficients $a_{11} = [2, 4]$ and $a_{12} = [-2, 1]$. The values of the left-hand side of the first equation L_1 at corner points (shown in italics in the figure) are calculated. The intersection points of the line $a_{11}x_1 + a_{12}x_2 = \overline{b_1}$ with sides of the rectangle are also determined. Next, we determine a set consisting of the points that are relevant to us (for which the value L_1 belongs to the interval b_1). In the set we have five points:

$$S_1 = \{(2, -2), (4, -2), (2, 1), (3, 1), (4, 0)\},$$

which define a pentagon. The area of the pentagon $AR_{1hit}(x_1, x_2) = 5.5$, and the area $AR_1 = 6$, hence the robustness of the first equation at the point $(x_1, x_2) = (-1, 1)$ is

$$r_1(x_1, x_2) = \frac{AR_{1hit}(x_1, x_2)}{AR_1} = \frac{11}{12}.$$

Figure 16 shows a rectangle defined by intervals of the coefficients $a_{21} = [-1, 2]$ and $a_{22} = [2, 4]$. The

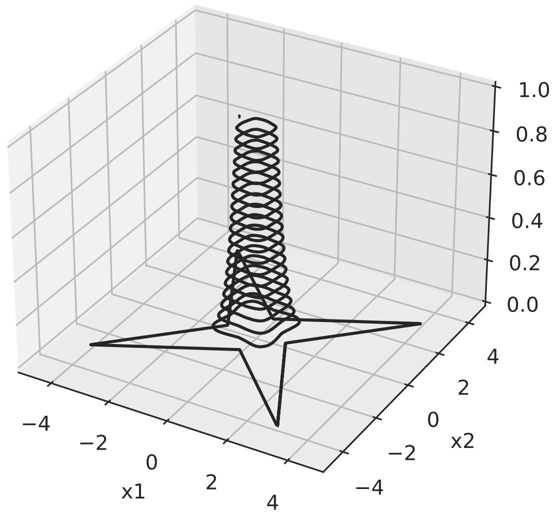

Fig. 17. 3D plot of the robustness $r(x_1, x_2)$ of the system (40).

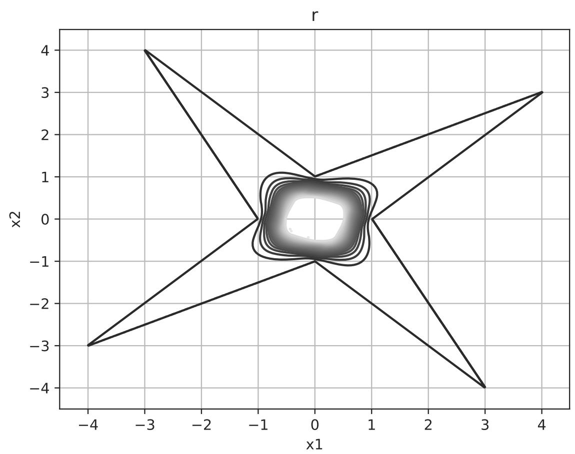

Fig. 18. Contour plot of the robustness $r(x_1, x_2)$ of the system (40).

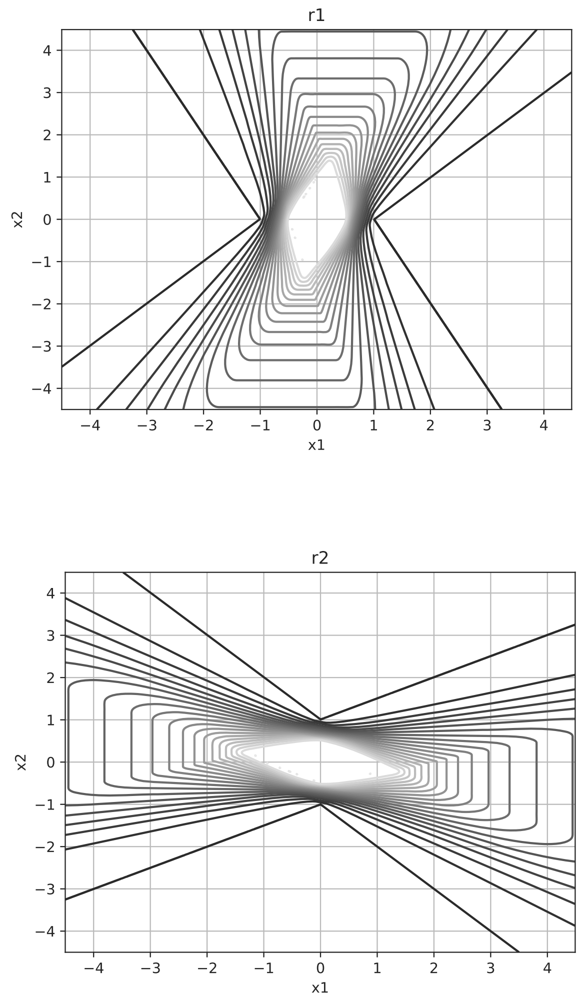

Fig. 19. Contour plots of the robustness $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ of the system (40).

values of the left-hand side of the second equation L_2 at corner points (shown in italics in the figure) are calculated. The intersection points of the line $a_{21}x_1 + a_{22}x_2 = \overline{b_2}$ with sides of the rectangle are also determined. Next, we determine a set consisting of the points that are relevant to us (for which the value L_2 belongs to the interval b_2). In the set we have four points:

$$S_2 = \{(-1, 2), (2, 2), (-1, 4), (0, 4)\},$$

which define the quadrangle. The area of the quadrangle $AR_{2hit}(x_1, x_2) = 4$, and the area $AR_2 = 6$; hence the robustness of equation two at the point $(x_1, x_2) = (-1, 1)$ is

$$r_2(x_1, x_2) = \frac{AR_{2hit}(x_1, x_2)}{AR_2} = \frac{4}{6}.$$

We get

$$\begin{aligned} r(x_1, x_2) &= r_1(x_1, x_2) \cdot r_2(x_1, x_2) \\ &= \frac{11}{12} \cdot \frac{4}{6} = \frac{11}{18} \approx 0.611\,. \end{aligned}$$

Figure 17 shows a 3D plot of the robustness $r(x_1, x_2)$ of the system (40), Fig. 18 shows a contour plot of the robustness, and Fig. 19 shows contour plots of the robustness of the first and the second equation.

Robustness calculations can also be performed numerically. For each point of the possible solution (x_1, x_2), we must calculate the robustness of the first and the second equation. To determine the robustness of the first one, we can greedily check all combinations of values from the intervals a_{11}, a_{12} with a sufficiently small step, testing whether, when substituted into the equation, the value on the left-hand side belongs to the interval b_1 on the right-hand side. The number of combinations $(a_{11},$

a_{12}) that satisfy the equation, divided by the number of all combinations examined, will approximate the robustness $r_1(x_1, x_2)$ for the point (x_1, x_2). Similar calculations can be performed for the second equation, for which we check all combinations of values from the intervals a_{21}, a_{22}. The number of combinations (a_{21}, a_{22}) that satisfy the second equation, divided by the number of all combinations examined, will approximate the robustness $r_2(x_1, x_2)$ for the point (x_1, x_2). As before, we calculate the robustness of the system at the point (x_1, x_2) as a product $r(x_1, x_2) = r_1(x_1, x_2) \cdot r_2(x_1, x_2)$. Experiments showed that the accuracy of numerically determined robustness is very good.

4.2. Example 4.

Let us consider the system of interval equations presented by Shary (1995) for which the fully robust solution does not exist,

$$\begin{aligned} [1,2]\, x_1 + [-1,1]\, x_2 &= [1,3], \\ [-1,1]\, x_1 + [1,2]\, x_2 &= [1,3]. \end{aligned} \tag{58}$$

The coefficients can be written in the form

$$\begin{aligned} a_{11} &= 1 + \gamma_{a_{11}}, & b_1 &= 1 + 2\gamma_{b_1}, \\ a_{12} &= -1 + 2\gamma_{a_{12}}, & b_2 &= 1 + 2\gamma_{b_2}, \\ a_{21} &= -1 + 2\gamma_{a_{21}}, \\ a_{22} &= 1 + \gamma_{a_{22}}. \end{aligned}$$

The values of the main determinant of the system of equations,

$$\begin{aligned} \Delta = (1 + \gamma_{a_{11}})(1 + \gamma_{a_{22}}) \\ - (-1 + 2\gamma_{a_{12}})(-1 + 2\gamma_{a_{21}}), \end{aligned} \tag{59}$$

belong to the interval $[0, 5]$. The solution of the system (58) is

$$\begin{aligned} x_1 &= \frac{1}{\Delta}\Big[(1 + 2\gamma_{b_1})(1 + \gamma_{a_{22}}) \\ &\quad - (1 + 2\gamma_{b_2})(-1 + 2\gamma_{a_{12}})\Big], \\ x_2 &= \frac{1}{\Delta}\Big[(1 + 2\gamma_{b_2})(1 + \gamma_{a_{11}}) \\ &\quad - (1 + 2\gamma_{b_1})(-1 + 2\gamma_{a_{21}})\Big]. \end{aligned} \tag{60}$$

The set of possible solutions is shown in Fig. 20.

Robustness calculations confirmed that the system does not have fully robust (tolerable) solutions. Thus, for the set of points representing possible (united) solutions, partial robustness calculations were performed. Figure 21 shows the robustness plot for possible solutions of the system. Robustness calculations were performed for each point described by coordinates (x_1, x_2). In the research, calculations were performed for a predetermined rectangle for which $x_1 \in [-4, 8]$ and $x_2 \in [-4, 8]$.

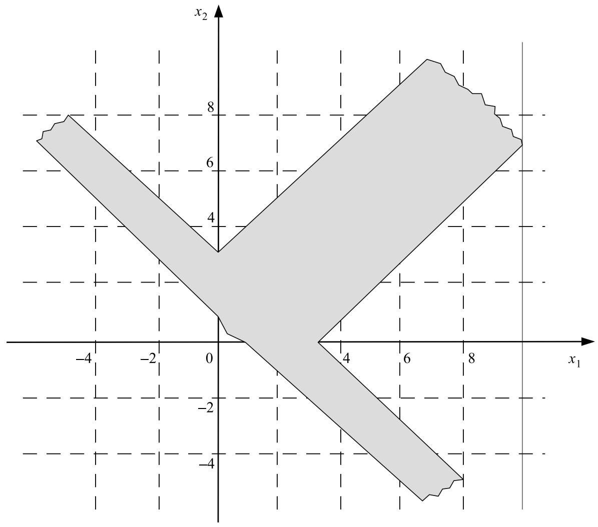

Fig. 20. Set of possible (united) solutions of the system (58).

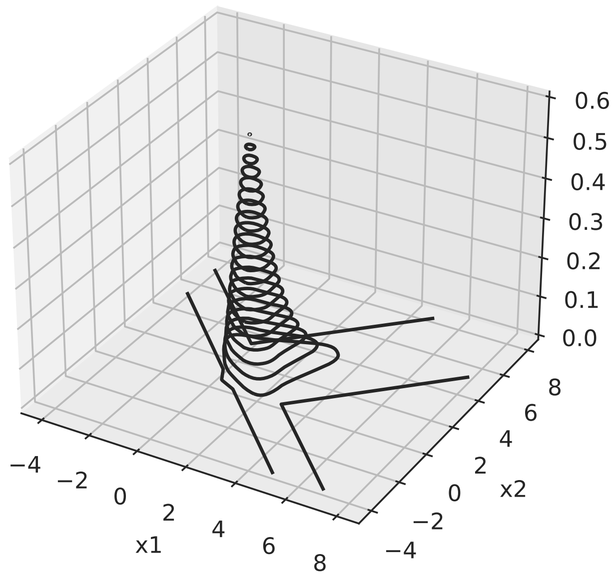

Fig. 21. 3D plot of the robustness $r(x_1, x_2)$ of the system (58).

These ranges were selected with the help of preliminary experiments. Calculations were performed for values x_1 and x_2 changing with a step of 0.01, i.e., 1201^2 points were examined. The maximum robustness of the system was found at the point $(x_1, x_2) = (1.11, 1.11)$:

$$r(1.11, 1.11) \approx 0.6006.$$

4.3. Example 5.

Let us consider the system of interval equations presented by Piegat and Pluciński (2017) for

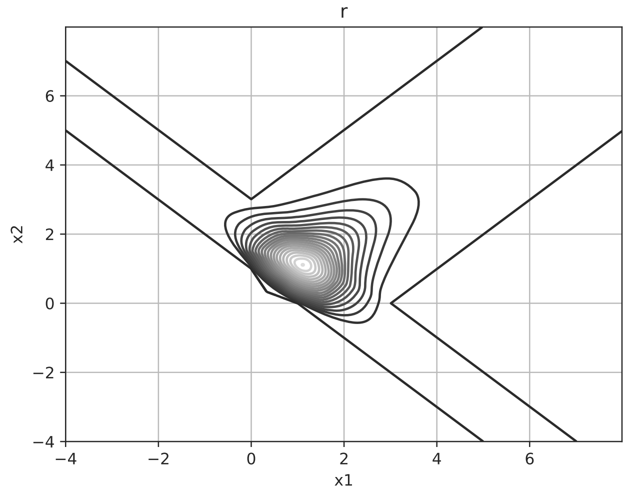

Fig. 22. Contour plot of the robustness $r(x_1, x_2)$ of the system (58).

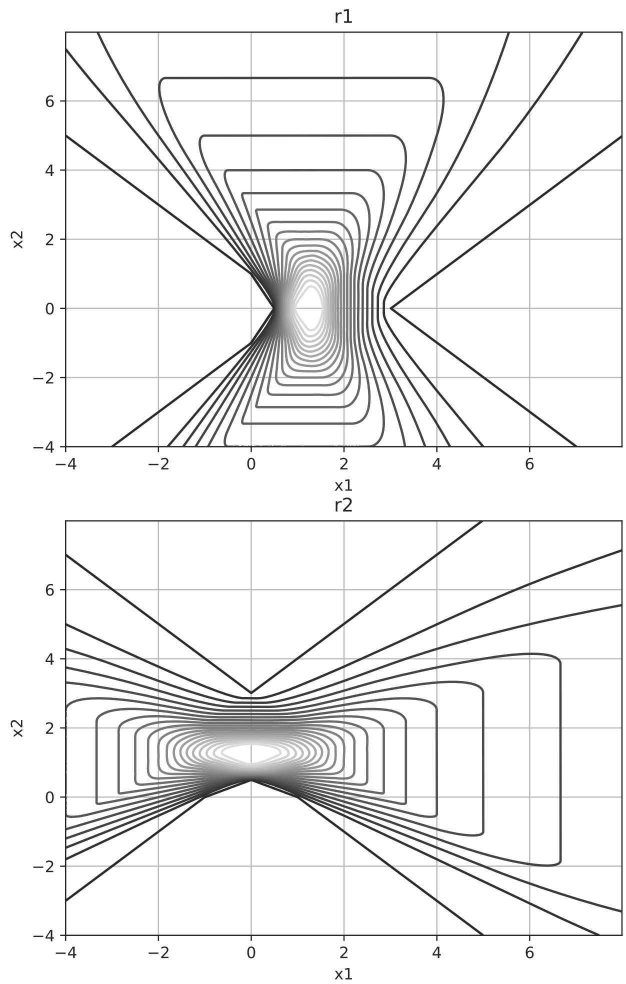

Fig. 23. Contour plots of the robustness $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ of the system (58).

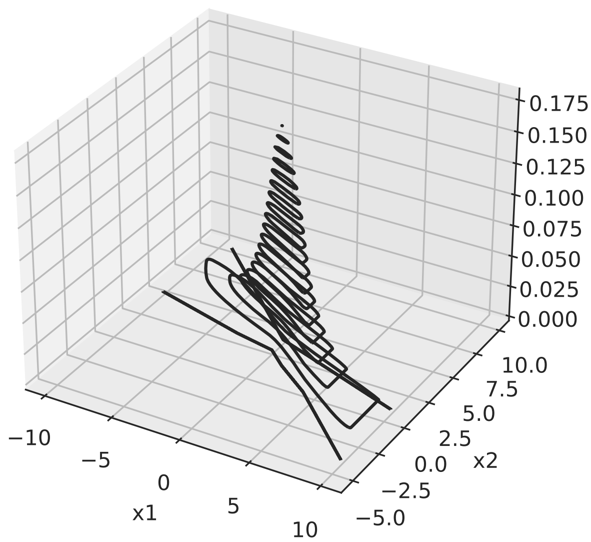

Fig. 24. 3D plot of the robustness $r(x_1, x_2)$ of the system (61).

which the fully robust solution does not exist,

$$\begin{aligned} &[1, 4]\, x_1 + [3, 6]\, x_2 = [15, 17], \\ &[2, 5]\, x_1 + [7, 9]\, x_2 = [21, 25]. \end{aligned} \tag{61}$$

The coefficients can be written in the form

$$\begin{aligned} a_{11} &= 1 + 3\gamma_{a_{11}}, & b_1 &= 15 + 2\gamma_{b_1}, \\ a_{12} &= 3 + 3\gamma_{a_{12}}, & b_2 &= 21 + 4\gamma_{b_2}, \\ a_{21} &= 2 + 3\gamma_{a_{21}}, \\ a_{22} &= 7 + 2\gamma_{a_{22}}. \end{aligned}$$

The values of the main determinant of the system of equations,

$$\begin{aligned} \Delta = {} & (1 + 3\gamma_{a_{11}})(7 + 2\gamma_{a_{22}}) \\ & - (3 + 3\gamma_{a_{12}})(2 + 3\gamma_{a_{21}}), \end{aligned} \tag{62}$$

belong to the interval $[-23, 30]$. The solution of the system (61) is

$$\begin{aligned} x_1 &= \frac{1}{\Delta}\Big[(15 + 2\gamma_{b_1})(7 + 2\gamma_{a_{22}}) \\ &\quad - (21 + 4\gamma_{b_2})(3 + 3\gamma_{a_{12}})\Big], \\ x_2 &= \frac{1}{\Delta}\Big[(21 + 4\gamma_{b_2})(1 + 3\gamma_{a_{11}}) \\ &\quad - (15 + 2\gamma_{b_1})(2 + 3\gamma_{a_{21}})\Big]. \end{aligned} \tag{63}$$

The set of possible solutions can be seen on the contour plot of the robustness (Fig. 25).

Robustness calculations confirmed that the system does not have fully robust (tolerable) solutions. Thus, for the set of points representing possible (united) solutions,

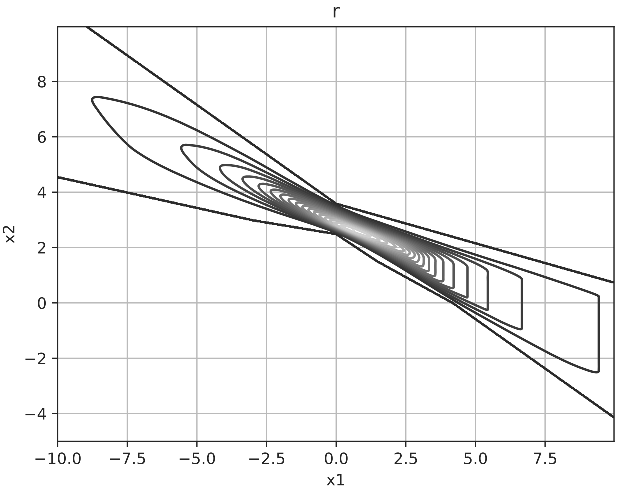

Fig. 25. Contour plot of the robustness $r(x_1, x_2)$ of the system (61).

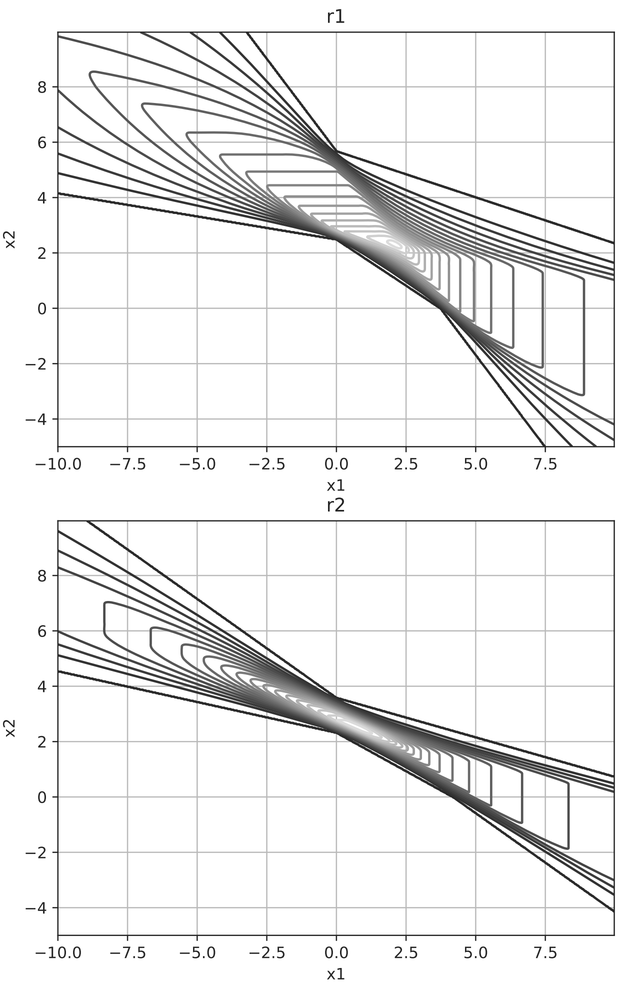

Fig. 26. Contour plots of the robustness $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ of the system (61).

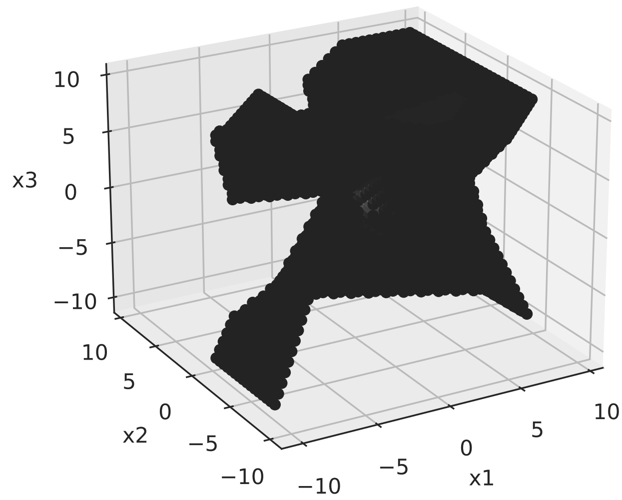

Fig. 27. 3D plot of the set of possible (united) solutions of the system (64).

partial robustness calculations were performed. Figure 24 shows the robustness plot for possible solutions of the system. The maximum robustness of the system was found at the point $(x_1, x_2) = (0.625, 2.625)$:

$$r(0.625, 2.625) \approx 0.1802.$$

4.4. Example 6. Let us consider the 3rd order system of interval equations,

$$\begin{aligned} [1,2]\, x_1 + [-2,-1]\, x_2 + [-1,1]\, x_3 &= [1,5], \\ [0,2]\, x_1 + [1,2]\, x_2 + [-1,1]\, x_3 &= [2,6], \\ [-1,1]\, x_1 + [0,2]\, x_2 + [1,2]\, x_3 &= [0,5]. \end{aligned} \tag{64}$$

Due to the higher order of the system, robustness calculations were carried out numerically. Figure 27 shows a 3D plot of points with robustness greater than 0, i.e., the set of possible (united) solutions. The maximum robustness of the system was found at the point $(x_1, x_2, x_3) = (2.21, 0.75, 0.74)$:

$$r(2.21, 0.75, 0.74) \approx 0.60576.$$

5. Conclusions

The paper presented a new method for determining robust, realistic, tolerant solutions of ILE systems, which enables solving ILE systems unsolvable so far. These enhanced solving possibilities were obtained by using MIA and by applying the concept of the robustness of the possible solutions. In practical problems, it is usually not possible to obtain the full robustness of the result to data uncertainty. Most often only the partial robustness can be obtained. The realistic approach is based on the practical application of solutions with high, though not perfect, robustness, i.e., on applying what is possible.

The article presented two methods of determining the robustness of possible solutions for data uncertainty: the exact geometric method and the slightly less accurate but practical numerical method. The proposed method for tolerant solving of ILE systems was explained with a few examples of different dimensions, including the benchmark ILE system (40) well known in the literature. This allowed a new, fuller view of this benchmark. In the opinion of the authors, the presented method allows relatively easy determining of the robustness of ILE systems due to the possibility of separate determination of the robustness of individual equations included in the system (problem decomposition).

Further research will focus on developing methods for determining solutions of a tolerant 3rd degree (and higher) static equation with the interval coefficients $[\underline{a}_3, \overline{a}_3]x^3 + [\underline{a}_2, \overline{a}_2]x^2 + [\underline{a}_1, \overline{a}_1]x + [\underline{a}_0, \overline{a}_0] = [\underline{y}_T, \overline{y}_T]$. It should be noted that the dimensionality of the problem and the difficulty of solving it increase as the degree of the equation increases. There is also work in progress on solving tolerance equations with coefficients in the form of fuzzy intervals.

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