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SPIKE PATTERNS AND CHAOS IN A MAP–BASED NEURON MODEL

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The work studies the well-known map-based model of neuronal dynamics introduced in 2007 by Courbage, Nekorkin and Vdovin, important due to various medical applications. We also review and extend some of the existing results concerning β-transformations and (expanding) Lorenz mappings. Then we apply them for deducing important properties of spike-trains generated by the CNV model and explain their implications for neuron behaviour. In particular, using recent theorems of rotation theory for Lorenz-like maps, we provide a classification of periodic spiking patterns in this model.

Keywords: neuronal dynamics, β-transformations, Farey–Lorenz permutations, periodic spiking, chaos.

1. Introduction

Courbage *et al.* (2007) proposed the following two-dimensional discrete neuron model:

$$\begin{aligned} x_{n+1} &= f_1(x_n, y_n) \\ &= x_n + F(x_n) - y_n - \beta H(x_n - d), \qquad \text{(1a)} \\ y_{n+1} &= f_2(x_n, y_n) = y_n + \varepsilon(x_n - J). \qquad \text{(1b)} \end{aligned}$$

The variables x and y stand, respectively, for the membrane potential of the neuron and the so-called recovery variable, aggregating the dynamics of all outward ionic currents (x_n and y_n describe the values of x and y, respectively, at consecutive time instances). Parameter J can be considered as a constant external stimulus and $\varepsilon > 0$ sets the time scale of the recovery variable. Parameters $\beta > 0$ and $d > 0$, in turn, control the threshold properties of oscillations. The function $H(x)$ is the usual Heaviside step function:

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and $F(x)$ is a piecewise linear continuous function defined as follows:

$$F(x) = \begin{cases} -m_0 x & \text{if } x \leq J_{\min}, \\ m_1(x-a) & \text{if } J_{\min} \leq x \leq J_{\max}, \\ -m_0(x-1) & \text{if } x \geq J_{\max}, \end{cases} \qquad (2)$$

where $m_0, m_1 > 0$, $1 > a > 0$, $J_{\min} = am_1/(m_0+m_1)$ and $J_{\max} = (m_0 + am_1)/(m_0+m_1)$.

The following assumptions are considered: $m_0 < 1$ and hence $\det \frac{\partial(f_1,f_2)}{\partial(x,y)} = 1 + F'(x) + \epsilon > 0$ for any $\epsilon \geq 0$, which guarantees the map to be one to one; lastly, the region of study is restricted to $0 < J < d$, $J_{\min} < d < J_{\max}$ resulting in $F'(d) > 0$ that is crucial in order to form chaotic behavior of the map.

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It may be worth pointing out that in the later alternative version of the model (Courbage and Nekorkin, 2010), the piecewise linear function F is replaced with a cubic-shape function (e.g., $F(x) = x(x-a)(1-x)$). It is due to the fact that the model is actually based on the FitzHugh–Nagumo model (FitzHugh, 1955) given by two ODEs with a third-degree polynomial in the first equation. The Courbage–Nekorkin–Vdovin (CNV for short) model is an example of so-called map-based (or discrete) models since it is given by the iterates of some map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (f_1(x,y), f_2(x,y))$ (see also the review articles by Courbage and Nekorkin (2010) or Ibarz *et al.* (2011)).

The CNV model appeared in many theoretical and data-based later studies. Its relative simplicity makes it possible to analytically or partially analytically study small ensembles of interacting neurons when each neuron is modelled by the CNV system. For instance, Courbage *et al.* (2012) consider two-interacting neurons of the type 1. However, such small networks can also represent larger ensembles of neurons with synchronized subpopulations as in the modelling of respiratory neural networks, reproducing some key experimental findings. See, for example, the work of Hess *et al.* (2013), where a mathematical model of respiratory rhytmogenesis is studied with the use of the piecewise linear CNV model refined to incorporate post-inhibitory rebound bursting oscillations.

One of the prominent applications of the CNV model in medical sciences is the work of Yu *et al.* (2016) on examining and modelling the interplay between the brainstem automatic network and cortical voluntary command on controlling the breathing process, relevant in numerous respiratory diseases, including the chronic obstructive pulmonary disease. In this study hypotheses drawn from the functional MRI imaging of patients were mathematically validated on the model of a neural network with the piecewise linear CNV model for each neural cluster. In turn, the work by Maslennikov and Nekorkin (2012) builds a discrete model of the olivo-cerebellar system based on the CNV model with cubic $F(x)$.

Apart from numerous applications, the model is still very challenging and attractive for analytical studies. In particular, Maslennikov and Nekorkin (2013) analyze the boundary crisis and transient chaos in the one-dimensional model (1a) with the cubic function $F(x)$ and a slowly varying control parameter y_n: $y_{n+1} = y_n + \varepsilon$. Transient chaos in a version of the one-dimensional model with periodic forcing is described by Maslennikov *et al.* (2018). In turn Yue *et al.* (2017) study a two-dimensional piecewise linear model with time-varying (pulse) external stimulus and claim that the CNV model could capture fundamental behaviours of the information processing of most conductance-based models.

For system (1), it turns out that the map describing the dynamics of the voltage variable x while the second variable y is kept constant (which we denote later by g in (6)) is a Lorenz-like map for a large range of parameter values of interest. Therefore, the first sections of the paper recall and develop some essential results on Lorenz-like maps which will be further applied directly to this model and its one-dimensional reduction. This shows how advanced results in discrete low-dimensional dynamical systems might be used for studying map-based neuron models. Another example in this vein is our recent work (Llovera-Trujillo *et al.*, 2023) on the Chialvo model which takes advantage of the theory of S-unimodal interval maps.

Apart from understanding and modelling biological neurons, another important field of research which develops rapidly in recent decades are artificial neural networks where the dynamics of single neurons is simple but the architecture of the network allows us to solve complex computational tasks and model dynamics of real systems (Korbicz *et al.*, 1999; Patan *et al.*, 2008).

The paper is organized as follows. Sections 2 and 3 contain basic definitions and facts on Lorenz maps and rotation theory. In turn, in Section 4 we present key properties of finite unions of periodic orbits. Sections 5 and 6 are devoted to the study of chaotic properties of Lorenz maps and β-transformations. In Section 7 we introduce the one-dimensional CNV model. Our main results concerning the plCNV model (piecewise linear case) are stated and proved in Sections 8. Finally, Sections 9 and 10 contain interpretation of our results in terms of spiking patterns and implications of the model for neuron behaviour.

2. Lorenz-like and expanding Lorenz maps

2.1. Chaos and its components. Let (X,d) be a compact metric space and $f\colon X \to X$ a function (not necessarily continuous). As the condition will recur, a set $U \subset X$ will be called *opene* when it is open and nonempty. Recall that f is called

- *transitive* if for every two opene sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$,
- *mixing* if for every two opene sets $U, V \subset X$ there is $N \in \mathbb{N}$ such that for every $n \geq N$ we have $f^n(U) \cap V \neq \emptyset$,
- *sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every neighbourhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$,
- *expansive* if there exists $\delta > 0$ such that for any $x, y \in X$, $x \neq y$, there exists $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$.

Remark 1. By definition, mixing implies transitivity and expansiveness implies sensitivity.

A function $f\colon X \to X$ is called *chaotic in the sense of Devaney* on X if

1. f is transitive,
2. the set of periodic points of f is dense in X,
3. f is sensitive.

For notational simplicity, we will formulate the definitions and results below for the unit interval $[0,1]$. However, all definitions make sense and all results still hold if we replace the unit interval by $[a,b]$ for fixed a and b. Namely, consider the linear change of variables (conjugacy) $h\colon [0,1] \to [a,b]$ given by $h(t) = a+t(b-a)$ and define $\widetilde{f}$ using the following diagram:

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "h"'] & {[0,1]} \arrow[d, "h"] \\ {[a,b]} \arrow[r, "\widetilde{f}"] & {[a,b]} \end{tikzcd} \tag{3}$$

Note that the condition $f(t) = x$ translates to $\widetilde{f}(h(t)) = h(x)$. Moreover, $f'(t) = \widetilde{f}'(h(t))$ if f is differentiable at $t \in [0,1]$ and, in consequence, both monotonicity and slope are preserved by h. Finally, if f is a Lorenz-like map (expanding Lorenz map, β-transformation) according to the definitions given below then so is $\widetilde{f}$ and vice versa.

2.2. Topological entropy.

For a piecewise continuous piecewise monotone map $f\colon [0,1] \to [0,1]$ we usually define *topological entropy* using the formula

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \ln c_n(f),$$

where $c_n(f)$ denotes the number of *laps* of f^n. Recall that the *lap* of f is the maximal interval, on which f is simultaneously monotone and continuous.

2.3. Lorenz-like maps.

A *Lorenz-like map* is a map f of an interval $[0,1]$ to itself, for which there exists a point $c \in (0,1)$ such that

- f is continuous and increasing (not necessarily strictly) on $[0,c)$ and on $(c,1]$,
- $\lim_{x\to c^-} f(x) = 1$ and $\lim_{x\to c^+} f(x) = f(c) = 0$.

Set $I_L = [0,c)$ and $I_R = [c,1]$. If $f(0) > f(1)$, that is, $f(I_L) \cap f(I_R) = \emptyset$, the Lorenz-like map f is called *nonoverlapping*. If $f(0) \le f(1)$, that is, $f(I_L) \cap f(I_R) \ne \emptyset$, the map is called *overlapping*.

2.4. Expanding Lorenz maps.

An *expanding Lorenz map* is a map $f\colon [0,1] \to [0,1]$ satisfying the following three conditions:

- there is a *critical point* $c \in (0,1)$ such that f is continuous and strictly increasing on $[0,c)$ and $(c,1]$,
- $\lim_{x\to c^-} f(x) = 1$ and $\lim_{x\to c^+} f(x) = f(c) = 0$,
- f is differentiable for all points not belonging to a finite set $\Omega \subset [0,1]$ and such that $\inf\{f'(x) \mid x \in [0,1]\setminus\Omega\} > 1$.

By definition, expanding Lorenz maps are overlapping Lorenz-like. Moreover, the assumption on the derivative yields expansiveness.

Proposition 1. *Every expanding Lorenz map is expansive.*

Proof. Let f be an expanding Lorenz map and $\beta = \inf\{f'(x) \mid x \in [0,1]\setminus\Omega\} > 1$. By definition, there exists $\delta_1 > 0$ such that $f(x) > 2/3$ for all $x \in [c-\delta_1, c)$ and $f(x) < 1/3$ for all $x \in [c, c+\delta_1]$. Let $\delta = \min\{1/3, \delta_1\}$. Assume that $x \ne y$ and $|x-y| < \delta$. Now suppose, contrary to our claim, that for all $n \in \mathbb{N}$ we have

$$d_n := |f^n(x) - f^n(y)| \le \delta. \tag{4}$$

Consider two cases:

- If $f^n(x), f^n(y) \in I_L$ or $f^n(x), f^n(y) \in I_R$ for all $n \in \mathbb{N}$, then, by the mean value theorem, $d_n \ge \beta d_{n-1} \ge \beta^n |x-y|$, which contradicts (4).
- If $f^n(x) \in I_L$ and $f^n(y) \in I_R$ for some $n \in \mathbb{N}$ (or vice versa), then $f^n(x) \in [c-\delta_1, c)$ and $f^n(x) \in [c, c+\delta_1]$ and, in consequence, $\left|f^{n+1}(x) - f^{n+1}(y)\right| > 1/3$, which also contradicts (4). ∎

3. Rotation number and interval

In this and the next section we mainly recall essential definitions and results of Geller and Misiurewicz (2018), which we will use in our analysis of the CNV model. Let f be a Lorenz-like map.

Definition 1. For a point $x \in [0,1]$ and a positive integer n we will denote by $R(x,n)$ the number of integers $i \in \{0,\dots,n-1\}$ such that $f^i(x) \in I_R$. If the limit

$$\rho(x) = \lim_{n\to\infty} \frac{R(x,n)}{n}$$

exists, we will call it the *rotation number* of x.

Remark 2. By definition, $0 \le \rho(x) \le 1$ if it exists.

Observe that it is extremely easy to compute the rotation number of a periodic point.

Proposition 2. *If x is a periodic point of f of period p then $\rho(x)$ exists and is equal to $R(x,p)/p$.*

The following classical result characterizes the uniqueness of the pointwise rotation number.

Theorem 1. (Rhodes–Thompson) *If $f(0) \geq f(1)$ (in particular, if the map is nonoverlapping), then all points have the same rotation number.*

In that case we will denote it by $\rho(f)$.

Let f be a Lorenz-like map. If $t \in f(I_L) \cap f(I_R)$, then we define the *water map* at level t by

$$f_t(x) = \begin{cases} \max(t, f(x)) & \text{if } x \in I_L, \\ \min(t, f(x)) & \text{if } x \in I_R. \end{cases}$$

It is obvious that this map is also Lorenz-like and $f_t(0) = f_t(1)$. Consequently, for fixed t, all points have the same rotation number $\rho(f_t)$ for it. It is known that $\rho(f_t)$ is an increasing continuous function of t, and if $f(0) \leq f(1)$, then the set of rotation numbers for f of all points having rotation number is equal to the interval $\left[\rho(f_{f(0)}), \rho(f_{f(1)})\right]$ (for details, see, e.g., Alsedà *et al.*, 1989). We will call it the *rotation interval* of f and denote it by $\mathrm{Rot}(f)$.

4. Finite unions of periodic orbits and Farey–Lorenz permutations

Let f be a Lorenz-like map. Below we summarize important properties of Farey–Lorenz permutations that allow us to deduce about the existence of periodic orbits with given itineraries. By the itinerary of a periodic orbit $\{x, f(x), \ldots, f^{q-1}(x)\}$ we mean a sequence of symbols L and R of length q such that the i-th element of this sequence ($i = 1, \ldots, q$) is L if $f^i(x) \in I_L$ and R otherwise (the sequence is given up to cyclic permutation). Proofs of Proposition 3 as well as Theorems 2 and 3 can be found in the work of Geller and Misiurewicz (2018).

4.1. Finite unions of periodic orbits (fupos). A finite union of periodic orbits of f is called, by acronym, a *fupo*. For each fupo we will consider its permutation, that is, if a fupo P consists of points $x_1 < \cdots < x_n$, and $f(x_i) = x_{\sigma(i)}$ for $i = 1, \ldots, n$, then σ is the permutation of P.

4.2. L-permutations. Permutations of fupos of Lorenz-like maps have a specific form. Namely, if a fupo has $n > 1$ elements, then there exists $k \in \{1, \ldots, n-1\}$ such that σ is increasing on $\{1, \ldots, k\}$ and on $\{k+1, \ldots, n\}$. We will call such permutations (and the permutation of $\{1\}$) *L-permutations*. If our fupo is a periodic orbit, then its rotation number is $(n-k)/n$.

For every L-permutation σ there exists a Lorenz-like map f with the fupo P such that P has permutation σ. A *canonical model* can be built as the "connect the dots" map with the dots being the points $(x, f(x))$ with coordinates as in Table 1.

Among L-permutations there are some special ones, which look like cyclic permutations for circle rotations. A cyclic L-permutation σ of $\{1, \ldots, n\}$ will be called a *twist* permutation if $\sigma(1) > \sigma(n)$. Similarly, a periodic orbit with such a permutation will be called a *twist* orbit. It is easy to describe explicitly a twist permutation of $\{1, \ldots, n\}$ with rotation number j/n. Namely,

$$\sigma(i) = i + j \mod n.$$

4.3. Farey–Lorenz permutations. An L-permutation σ of $\{1, \ldots, p+q\}$ will be called a *Farey–Lorenz permutation* (or *FL-permutation*) if σ consists of two cycles, both of them twist, of period p and q, with rotation numbers a/p and b/q respectively, and $a/p < b/q$ are Farey neighbours, that is, $bp = aq + 1$. An example of a fupo of FL-type (similar to those occurring in the plCNV model) is presented in Fig. 1.

Let us assume that σ is an FL-permutation with cycles of rotation numbers a/p and b/q, where $a/p < b/q$ are Farey neighbours and $p < q$. We will call those cycles *slow* and *fast*, respectively. Note that (Geller and Misiurewicz, 2018):

- the slow cycle contains 1, and the fast cycle contains $p+q$,
- the fast cycle contains 2.

We will refer to P as the slow orbit and Q as the fast orbit. Let the points of P be $x_1 < x_2 < \cdots < x_p$ and the points of Q, $y_1 < y_2 < \cdots < y_q$. Write $J_1 = [x_1, y_1]$ and $J_j = [y_{j-1}, y_j]$ for $j = 2, \ldots, q$.

Proposition 3. *With the notation we adopted,*

$$f(x_i) = x_{i+a \mod p} \quad \textit{and} \quad f(y_j) = y_{j+b \mod q}.$$

The relative order of the points of the orbits P and Q is given by the following rule: $x_1 < y_1$; then for $i = 1, \ldots, p-1$, if $j = 1 + ia \mod p$ and $l = ib \mod q$, then $y_{l-1} < x_j < y_l$.

Theorem 2. *Let g be a Lorenz-like map. If a number r/s is in the rotation interval of g, and r, s are coprime, then g has a twist periodic orbit of period s and rotation number r/s.*

Proposition 4. *If g is an expanding Lorenz map and S is a periodic L-R sequence, then there is at most one periodic orbit with itinerary S.*

Proof. Assume we have two distinct q-periodic orbits $\{x_1 < \cdots < x_q\}$ and $\{y_1 < \cdots < y_q\}$ with the same itinerary. Hence, by the mean value theorem, $|f^q(x_1) - f^q(y_1)| > |x_1 - y_1|$, a contradiction. ■

Table 1. “Connect the dots” map.

x	0	$\frac{1}{n+1}$	$\dots$	$\frac{k}{n+1}$	$\frac{k+\frac{1}{2}}{n+1}$	$\frac{k+\frac{1}{2}}{n+1}$	$\frac{k+1}{n+1}$	$\dots$	$\frac{n}{n+1}$	1
$f(x)$	$\frac{\sigma(1)}{n+1}$	$\frac{\sigma(1)}{n+1}$	$\dots$	$\frac{\sigma(k)}{n+1}$	1	0	$\frac{\sigma(k+1)}{n+1}$	$\dots$	$\frac{\sigma(n)}{n+1}$	$\frac{\sigma(n)}{n+1}$

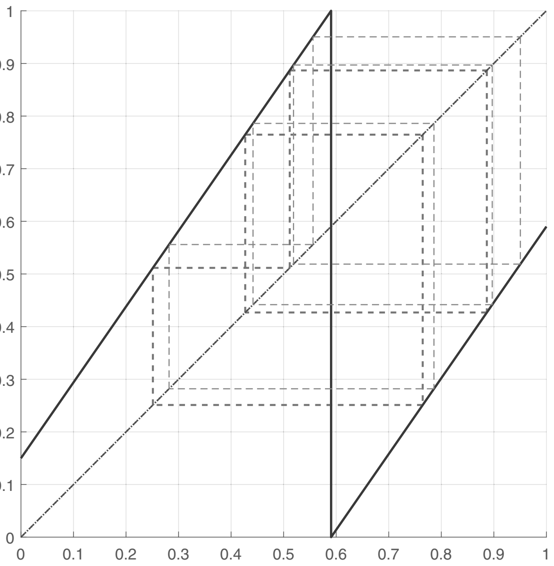

Fig. 1. Fupo of FL-type $\left(\frac{2}{5} < \frac{3}{7}\right)$.

Corollary 1. *If g is an expanding Lorenz map and r/s is in the rotation interval of g (r and s coprime) then there is only one twist periodic orbit with rotation number r/s.*

Corollary 2. *If g is an expanding Lorenz map, then there is at most $\binom{q}{p}/q$ periodic orbits with rotation number p/q (q, p coprime).*

Theorem 3 below follows from Theorem 2 of Geller and Misiurewicz (2018) and considerations in the second paragraph of Section 5 therein.

Theorem 3. *Let g be a Lorenz-like map. Assume that the rotation interval of g contains the interval $[a/p, b/q]$, where $a/p < b/q$ are Farey neighbours, and $p < q$. Then g has a twist periodic P orbit of period p and rotation number a/p and a twist periodic orbit Q of period q and rotation number b/q. The union of these orbits forms a fupo with the FL-permutation. Moreover, there exist periodic orbits with itineraries being concatenations of finitely many periodic itineraries (starting at x_1 or y_1) of P and Q.*

5. Lorenz maps and chaos

We say that a map $f\colon [0,1] \to [0,1]$ is *strongly transitive* if for every nonempty open subinterval $J \subset (0,1)$ there exists $n \in \mathbb{N}$ such that $\bigcup_{i=0}^{n} f^n(J) \supset (0,1)$. Note that strong transitivity implies transitivity.

The following three results, which are essential for our considerations, can be found in the work of Oprocha *et al.* (2019, Thms. 4.5, 4.6, 4.7, 4.8), Kameyama (2002, Thm. 3) or Afraimovich and Hsu (2002, Thm. 3.1.1), respectively. Let

$$f_1(x) = \sqrt{2}x + \frac{2-\sqrt{2}}{2} \pmod 1,$$
$$f_2(x) = \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2} \pmod 1,$$
$$f_3(x) = \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2} \pmod 1.$$

for $x \in [0,1]$.

Theorem 4. *Let f be an expanding Lorenz map and $\beta = \inf\{f'(x) \mid x \in [0,1] \setminus \Omega\}$. If*

- *$\sqrt{2} \le \beta \le 2$ and $f \ne f_1$ or*
- *$\sqrt[3]{2} \le \beta < \sqrt{2}$ and $f(0) \ge \frac{1}{1+\beta}$ and $f \ne f_2$ or*
- *$\sqrt[3]{2} \le \beta < \sqrt{2}$ and $f(1) \le 1 - \frac{1}{1+\beta}$ and $f \ne f_3$,*

then f is mixing. Moreover, the maps f_1, f_2 and f_3 are transitive but not mixing.

Theorem 5. *Assume that $f\colon [0,1] \to [0,1]$ is piecewise continuous piecewise strictly monotone. If f is transitive then it is strongly transitive.*

Theorem 6. *If an expanding Lorenz map is strongly transitive then the set of its periodic points is dense in $[0,1]$.*

Remark 3. Theorem 6 has been originally formulated for the so-called *Lorenz-type* maps, but it is easy to check that expanding Lorenz maps are also Lorenz-type.

It is well known that for continuous maps on intervals transitivity implies chaos. As we see below, a similar result holds for expanding Lorenz maps.

Proposition 5. *An expanding Lorenz map is transitive if and only if it is chaotic in the sense of Devaney on $[0,1]$.*

Proof. Chaos obviously implies transitivity, so we only need to show the opposite implication. By Theorem 5, each transitive expanding Lorenz map is strongly transitive. Hence, by Theorem 6, the set of its periodic points is dense in $[0,1]$. It remains to prove the sensitive dependence on initials conditions, but, by Proposition 1, our map is even expansive. ■

The following result provides a sufficient condition for Devaney's chaos for expanding Lorenz maps.

Theorem 7. *Let f be an expanding Lorenz map and $\beta = \inf\{f'(x) \mid x \in [0,1] \setminus \Omega\}$. If one of the following conditions is satisfied:*

(i) $\sqrt{2} \leq \beta \leq 2$,

(ii) $\sqrt[3]{2} \leq \beta < \sqrt{2}$ *and* $f(0) \geq \frac{1}{1+\beta}$,

(iii) $\sqrt[3]{2} \leq \beta < \sqrt{2}$ *and* $f(1) \leq 1 - \frac{1}{1+\beta}$,

then f is chaotic in the sense of Devaney on the interval $[0,1]$. Moreover,

- *f is strongly transitive and expansive,*
- *if $f \neq f_1$ and $f \neq f_2$ and $f \neq f_3$ then f is mixing.*

Proof. It is an immediate consequence of Theorem 4 and Proposition 5. Strong transitivity and expansiveness can be concluded from the proof of Proposition 5. ■

6. β-transformations

6.1. Definition. Let $1 < \beta \leq 2$, $\alpha \geq 0$ and $\alpha + \beta \leq 2$. The map $T\colon [0,1] \to [0,1]$ of the form

$$T(x) = \beta x + \alpha \pmod 1$$

is called a *β-transformation*.

Remark 4. Every β-transformation has a point of discontinuity at $c = (1-\alpha)/\beta$ and $T(c) = 0$. Moreover, $T(0) = \alpha$ and $T(1) = \alpha + \beta - 1$. Note that β-transformations are expanding Lorenz maps and Lorenz-like of a constant slope.

6.2. Acip. Here we summarize some metric (measure-theoretic) properties of β-transformations from Theorem 3 of Hofbauer (1979) and Theorem 2 of Hofbauer (1981).

Theorem 8. *The β-transformation T has topological entropy $h(T) = \ln \beta$ and unique T-invariant probability measure (acip for short) μ with maximal entropy, i.e., such that the metric entropy $h_\mu(T)$ is equal to the topological entropy $h(T)$. The measure μ is absolutely continuous with respect to Lebesgue measure. The density of μ is given by the formula*

$$K \sum_{n=0}^{\infty} \beta^{-n} \big(\chi_{[0,T^n(1)]}(x) - \chi_{[0,T^n(0)]}(x)\big),$$

where K is a normalizing factor. Moreover, the support of μ is a finite union of intervals and it is the whole interval $[0,1]$ if

(i) $\sqrt{2} \leq \beta \leq 2$ *or*

(ii) $1 < \beta < \sqrt{2}$ *and* $0 \leq \alpha \leq 1 - 1/\beta$ *or*

(iii) $1 < \beta < \sqrt{2}$ *and* $1 - \beta + 1/\beta \leq \alpha \leq 2 - \beta$.

Remark 5. Assume $1 < \beta \leq 2$. Observe that the acip μ of T is *equivalent to the Lebesgue measure* if and only if $\operatorname{supp}(\mu) = [0,1]$ (see the proof of Lemma 4.1 in the work of Ding *et al.* (2010)). Moreover, Parry (1979) proved that if $1 < \beta < \sqrt{2}$ and $c = 1/2$ (or equivalently $\alpha = 1 - \beta/2$) then μ is not equivalent to the Lebesgue measure. Since the measure μ is unique, the β-transformation T is *ergodic* with respect to μ, i.e., if $T^{-1}B = B$ for some Borel set $B \subset [0,1]$ then $\mu(B) = 0$ or $\mu(B) = 1$. Under some assumptions on β and α, the transformation T has even stronger metric properties (for details, see Palmer, 1979), but we will not study and use them in this paper.

6.3. Transitivity and chaos. The following result by Ding *et al.* (2010, Lem. 4.1.) provides a nice metric characterization of topological transitivity.

Proposition 6. *Let $1 < \beta \leq 2$ and μ be the acip of the β-transformation T. Then T is transitive if and only if $\operatorname{supp}(\mu) = [0,1]$.*

The following result concerning β-transformations of Oprocha *et al.* (2019, Thm. 7.1.) corresponds to Theorem 4 proven by the same authors for expanding Lorenz maps. However, it is worth pointing out that Theorem 4 gives a sufficient condition for mixing in a broader class of functions (expanding Lorenz maps) while Theorem 9 gives a necessary and sufficient condition for mixing in a narrower class (β-transformations).

Theorem 9. *Let T be a β-transformation and $\sqrt[3]{2} \leq \beta \leq 2$. Then T is mixing if and only if one of the following conditions is satisfied:*

- $\sqrt{2} \leq \beta \leq 2$ *and* $f \neq f_1$ *or*
- $\sqrt[3]{2} \leq \beta < \sqrt{2}$ *and* ($\alpha < \frac{1}{\beta(1+\beta)}$ *or* $2-\beta-\frac{1}{\beta(1+\beta)} < \alpha$) *and* $f \neq f_2$ *and* $f \neq f_3$.

Let us present the complete characterization of chaos parameters for β-transformations in the parameter region $\sqrt[3]{2} \leq \beta \leq 2$ and $0 \leq \alpha \leq 2 - \beta$.

Theorem 10. *Assume that T is a β-transformation and $\sqrt[3]{2} \leq \beta \leq 2$. Then T is chaotic in the sense of Devaney on the interval $[0,1]$ if and only if one of the following conditions is satisfied:*

(i) $\sqrt{2} \leq \beta \leq 2$,

(ii) $\sqrt[3]{2} \leq \beta < \sqrt{2}$ *and* ($\alpha < \frac{1}{\beta(1+\beta)}$ *or* $2 - \beta - \frac{1}{\beta(1+\beta)} < \alpha$).

Moreover, under these assumptions

- *T is strongly transitive and expansive,*
- *if $T \neq f_1$, $T \neq f_2$ and $T \neq f_3$ then T is mixing.*

Proof. Our claim can be shown in the same manner as Theorem 7 using Theorem 4, Proposition 5 and Theorem 9. ■

In the case $1 < \beta < \sqrt[3]{2}$ we can formulate a sufficient condition for chaos for β-transformations.

Theorem 11. *Let T be a β-transformation with $1 < \beta < \sqrt[3]{2}$. Then T is chaotic in the sense of Devaney on the interval $[0, 1]$ if*

- $0 \leq \alpha \leq 1 - 1/\beta$ *or*
- $1 - \beta + 1/\beta \leq \alpha \leq 2 - \beta$.

Proof. By Theorem 8, our assumptions guarantee that $\operatorname{supp}(\mu) = [0, 1]$, which is equivalent to transitivity of T from Proposition 6. Finally, by Proposition 5, transitivity implies chaos. ■

6.4. Fixed and periodic points.

The simplicity of the formula defining β-transformation allows us to easily derive the following two results concerning fixed points and 2-periodic orbits.

Lemma 1. *β-transformation $T(x)$ admits fixed points only when $\alpha = 0$ or when $\alpha + \beta = 2$. In the first case, i.e., $\alpha = 0$, there is a fixed point $x_{f_1} = 0$ and the rotation interval $\operatorname{Rot}(T)$ contains 0. In the second case, $\alpha + \beta = 2$, there is a fixed point $x_{f_2} = 1$ and the rotation interval contains 1. In particular, $\operatorname{Rot}(T) = [0, 1]$ only when $\alpha = 0$ and $\beta = 2$, in which case $T(x) = 2x \pmod 1$ is a standard chaotic dyadic transformation.*

The above lemma is a simple observation and does not require a proof. On the other hand, some short calculations lead to the following result.

Lemma 2. *Suppose that $\alpha > 0$ and $\alpha + \beta < 2$. If*

$$\tilde{\beta} < \beta < \min\Big\{2 - \alpha, \frac{1-\alpha}{\alpha}\Big\}, \tag{5}$$

where

$$\tilde{\beta} := \frac{1 - \alpha + \sqrt{(\alpha-1)(\alpha-5)}}{2},$$

then T admits a 2-periodic orbit and 2 is minimal among periods of periodic points of T. In particular, under these assumptions $1/2 \in \operatorname{Rot}(T)$ and $\{0, 1\} \cap \operatorname{Rot}(T) = \emptyset$.

Proof. Note that the requirements $\beta < (1 - \alpha)/\alpha$ and $1 < \beta$ imply that $\alpha < 1/2$ and consequently, that $\tilde{\beta} > 1$. Further, under these conditions none of the points 0, c and 1 acts as a fixed point of T and thus the map T does not have fixed points. If suffices to show that under (5) there exists a period-two point $z \in (0, c)$, i.e.,

$$0 < z < c \quad \text{and} \quad T^2(z) = z,$$

which is equivalent to

$$0 < z < \frac{1-\alpha}{\beta} \quad \text{and} \quad \beta^2 z + \beta\alpha + \alpha - 1 = z.$$

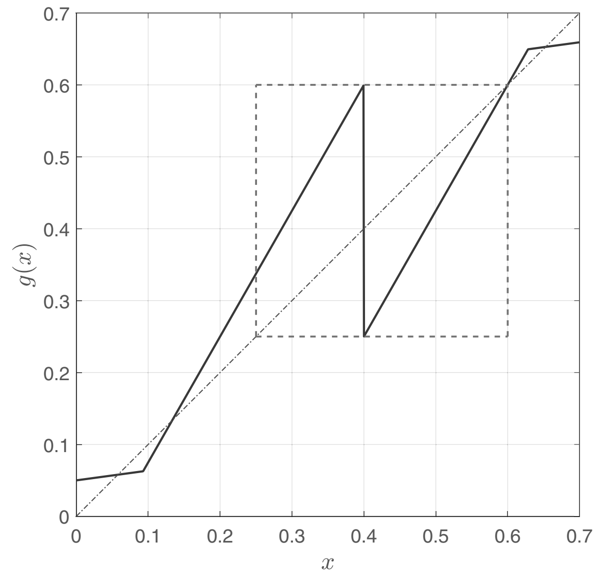

Fig. 2. Plot of the plCNV function and the invariant interval.

Solving the last equality for z gives

$$z := \frac{1}{\beta-1}\Big(\frac{1}{\beta+1} - \alpha\Big).$$

Now, it remains to solve with respect to β the following double inequality

$$0 < \frac{1}{\beta-1}\Big(\frac{1}{\beta+1} - \alpha\Big) < \frac{1-\alpha}{\beta}.$$

As $\beta > 1$, this yields

$$\beta < \frac{1-\alpha}{\alpha} \quad \text{and} \quad \beta^2 + \beta(\alpha - 1) + \alpha - 1 > 0.$$

The first inequality gives the upper bound for β in (5). On the other hand, since the discriminant $\Delta = (\alpha - 1)(\alpha - 5)$ in the second inequality is positive, this inequality is satisfied (considering the assumption $\beta > 1$) if

$$\beta > \frac{1 - \alpha + \sqrt{(\alpha-1)(\alpha-5)}}{2}.$$

■

We have now obtained a picture of periodic and chaotic properties for Lorenz maps. At the end of this theoretical part, let us mention that in their recent works Cholewa and Oprocha (2021a; 2021b) developed the theory of limit sets and renormalization for Lorenz maps.

7. One-dimensional Courbage–Nekorkin–Vdovin model

Now we will analyze more closely the 1D CNV model. When the variable $y = y_0$ is assumed to be constant, $0 <$

$J < d$ and $J_{\min} < d < J_{\max}$, the CNV model can be reduced to a one-dimensional map $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x + F(x) - y_0 - \beta H(x-d). \tag{6}$$

In general, in the above function we will vary the parameters β and/or y_0 keeping all other parameters fixed. One can consider two versions of the 1D CNV model:

- a *piecewise linear* case (plCNV for short), when F is defined by Eqn. (2),
- a *nonlinear* case (nlCNV for short), when $F(x) = \mu x(x-a)(1-x)$ with $0 < a < 1$ and $\mu \geq 1$.

Observe that in both the cases the function g is discontinuous at d and the plot of F has the shape of the upside down reversed $\boldsymbol{N}$ letter. Moreover, both maps exhibit quite similar dynamics. For that reason we will mainly focus on the piecewise linear map.

Let us have a more detailed look at the 1D plCNV model, which is our main object of interest in this paper. Note that in the piecewise linear case the map g can be re-written explicitly as

$$g(x) = \begin{cases} (1-m_0)x - y_0 & \text{if } x \leq J_{\min}, \\ qx - y_0 - am_1 & \text{if } J_{\min} < x < d, \\ qx - y_0 - am_1 - \beta & \text{if } d \leq x \leq J_{\max}, \\ (1-m_0)x - y_0 + m_0 - \beta & \text{if } x \geq J_{\max}, \end{cases}$$

where $q = 1 + m_1$. Recall that $J_{\min} = am_1/(m_0+m_1)$ and $J_{\max} = (m_0 + am_1)/(m_0+m_1)$. Let us assume that the parameters a, d, m_0, m_1 are fixed and satisfy the following conditions:

1. $0 < a < 1$,
2. $J_{\min} < d < J_{\max}$,
3. $0 < m_0 < 1$,
4. $0 < m_1 \leq 1$.

The first condition means that a is a zero of F, which lies between $J_{\min}$ and $J_{\max}$. The second one means that the discontinuity point of g also lies between these two points. The third and fourth conditions guarantee that two of the four possible fixed points of g are stable while the other two are unstable (if they exist). An illustrative plot of the function g involved in the plCNV model is provided in Fig. 2.

8. Periodicity and chaos in the 1D plCNV model

The main goal of this section is to study the dynamics of the plCNV map in the parameter plane (β, y_0). We work with the assumptions from Section 7. However, the further restriction of the parameter region allows us to obtain interesting dynamical properties of the plCNV map. Namely, we list the following additional assumptions:

Assumption A1: $\beta > \beta_0 = F(J_{\max}) - F(J_{\min})$,

Assumption A2: $\beta \leq \beta_1 = \min\{J_{\max} - d, d - J_{\min}\}$,

Assumption A3: $y_0 > F(d) - \beta$,

Assumption A4: $y_0 < F(d)$,

Assumption A5: $y_0 \geq F(d) - \beta/q$,

Assumption A6: $y_0 \leq F(d) - \beta(q-1)/q$,

Assumption A7: $y_0 > F(J_{\min})$,

Assumption A8: $y_0 < F(J_{\max}) - \beta$.

Let us explain thoroughly the geometric meaning of these assumptions. A special role in our considerations is played by an interval $I = [b, c] \subset \mathbb{R}$, where

$$b = \lim_{x \to d^+} g(x) = g(d) = qd - y_0 - am_1 - \beta,$$
$$c = \lim_{x \to d^-} g(x) = qd - y_0 - am_1.$$

Note that if β is positive then I is well defined and its length is equal to β.

Assumption A1 guarantees that $\beta > 0$ and that for a fixed β satisfying this condition there is a nonempty open interval $(F(J_{\max}) - \beta, F(J_{\min}))$ of values of y_0 for which the map g has no fixed points.

Assumption A3 reads $b < d$ and Assumption A4 reads $d < c$, so they both imply that the discontinuity point d lies inside $[b, c]$. Assume for a moment that Assumptions A3 and A4 hold. Now Assumption A2 implies that $J_{\min} \leq b$ and $J_{\max} \geq c$. Namely, then $d - b \leq c - b = \beta \leq d - J_{\min}$ and, in consequence, $J_{\min} \leq b$. Similarly, we justify that $J_{\max} \geq c$. This means that g restricted to $[b, c]$ is a constant slope map with the slope q.

Observe that, by the definitions of b and c, the interval $[b, c]$ is invariant, i.e., $g\big([b, c]\big) = [b, c]$ if and only if $g(b) \geq b$ and $g(c) \leq c$. This is true if Assumptions A2–A6 are satisfied. Namely, then $g(b) - b = g(b) - g(d) = q(b-d) + \beta = q(F(d) - y_0 - \beta(q-1)/q) \geq 0$ and, analogously, $g(c) - c = q(F(d) - y_0 - \beta/q) \leq 0$.

Furthermore, Assumption A5 (resp. A6) means that an unstable fixed point on the right (resp., left) branch of the plot of g is outside the interval $[b, c]$. Finally, Assumptions A7 and A8 are related to the existence of a stable fixed point for g (see Theorem 12 below).

The following summary result is an immediate consequence of the above considerations and simple calculations.

Theorem 12. (Fixed points)

- *Under Assumption A7 g has a stable fixed point $x_{\mathrm{ls}} = -y_0/m_0$ on the left branch of the plot.*
- *Under Assumption A4 and A7 g has an unstable fixed point $x_{\mathrm{lu}} = a + y_0/m_1$ on the left branch of the plot.*

- *Under Assumption A8 g has a stable fixed point $x_{\text{rs}} = 1 - (y_0 + \beta)/m_0$ on the right branch of the plot.*
- *Under Assumptions A3 and A8 g has an unstable fixed point $x_{\text{ru}} = a + (y_0 + \beta)/m_1$ on the right branch of the plot.*
- *Under Assumption $F(J_{\max}) - \beta < y_0 < F(J_{\min})$, g has no fixed points in $\mathbb{R}$.*

Remark 6. Moreover, observe that

- the existence of an unstable fixed point always implies the existence of a stable fixed point for g;
- if Assumption A1 is met, then the fixed points on the left branch of the plot of g cannot coexist with the fixed points on the right branch of the plot;
- if $y_0 = F(J_{\min})$ (resp. $y_0 = F(J_{\max}) - \beta$) then $J_{\min}$ (resp. $J_{\max}$) is a semistable fixed point and a fold bifurcation occurs at this parameter value;
- if $y_0 = F(d) - \beta/q$ (resp. $y_0 = F(d) - \beta(q-1)/q$) then the unstable fixed point on right (resp. left) branch of the plot coincides with the right (resp. left) end of the interval $[b, c]$.

For the rest of this section we make Assumptions A1–A6. Let G denote g restricted to $[b, c]$. Observe that G governs the dynamics of the 1D plCNV on the invariant interval $[b, c]$. The next proposition is an easy consequence of our assumptions.

Proposition 7. (β-transformation) *The map $G\colon [b, c] \to [b, c]$*

- *is a well-defined β-transformation with slope q and point of discontinuity d,*
- *has no fixed points in (b, c).*

Figure 3 presents a plot of the map G for the following choice of parameters in the plCNV model: $a = 0.2$, $d = 0.4$, $m_0 = 0.864$, $m_1 = 0.65$, $\beta = 0.35$ and $y_0 = -0.05$. These parameter values are exemplary but in the range considered by Courbage *et al.* (2007).

Now let us formulate the main results, which explain the behaviour of the 1D plCNV model in the invariant interval $[b, c]$. Recall that, by definition, $G(b) = qb - y_0 - am_1$ and $G(c) = qc - y_0 - am_1 - \beta$. Our first result provides the conditions for Devaney's chaos in the 1D plCNV model.

Theorem 13. (Chaos)

1. *Assume that $\sqrt[3]{2} \le q \le 2$. Then G is chaotic in the sense of Devaney on the interval $[b, c]$ if and only if one of the following conditions is satisfied:*
   - *(i)* $\sqrt{2} \le q \le 2$,
   - *(ii)* $\sqrt[3]{2} \le q < \sqrt{2}$ *and* $(G(b) < b + \frac{c-b}{q(1+q)}$ *or* $c - \frac{c-b}{q(1+q)} < G(c))$.

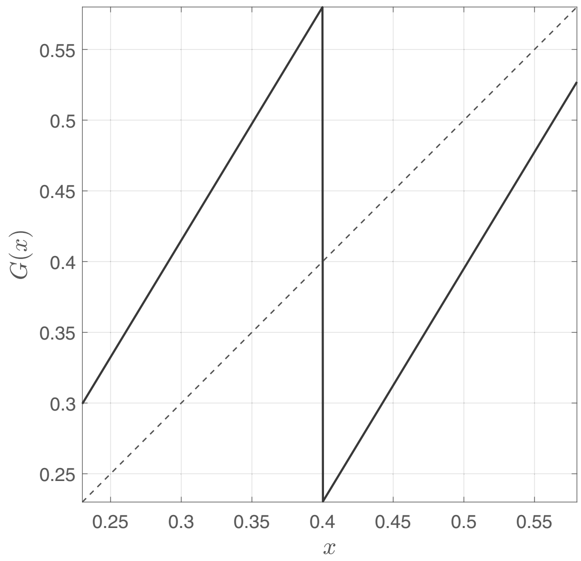

Fig. 3. Restriction of the plCNV function as a β-transformation.

2. *Assume that $1 < q < \sqrt[3]{2}$. Then G is chaotic in the sense of Devaney on the interval $[b, c]$ if one of the following conditions is satisfied:*
   - *(i)* $G(b) \le c - \frac{c-b}{q}$,
   - *(ii)* $b + \frac{c-b}{q} \le G(c)$.

Proof. Theorem 10 implies (1). Similarly, Theorem 11 implies (2). ■

The next result summarizes the metric properties of the restricted 1D plCNV model.

Theorem 14. (acip) *The map G has topological entropy $h(G) = \ln q$ and unique G-invariant probability measure μ with maximal entropy. The measure μ is absolutely continuous with respect to the Lebesgue measure. Moreover, the support of μ is a finite union of intervals and it is the whole interval $[b, c]$, which means that μ is equivalent to the Lebesgue measure, if*

- *(i)* $\sqrt{2} \le q \le 2$ *or*
- *(ii)* $1 < q < \sqrt{2}$ *and* $G(b) \le c - \frac{c-b}{q}$ *or*
- *(iii)* $1 < q < \sqrt{2}$ *and* $b + \frac{c-b}{q} \le G(c)$.

Proof. It is an immediate consequence of Theorem 8. ■

Finally, the parameter regions of the existence and stability of fixed points and chaos are depicted in Fig. 4.

9. Interpretation of orbit itineraries in terms of spiking patterns

Characterizing itineraries of periodic orbits is not only an interesting and challenging mathematical problem but it is also meaningful from the point of view of

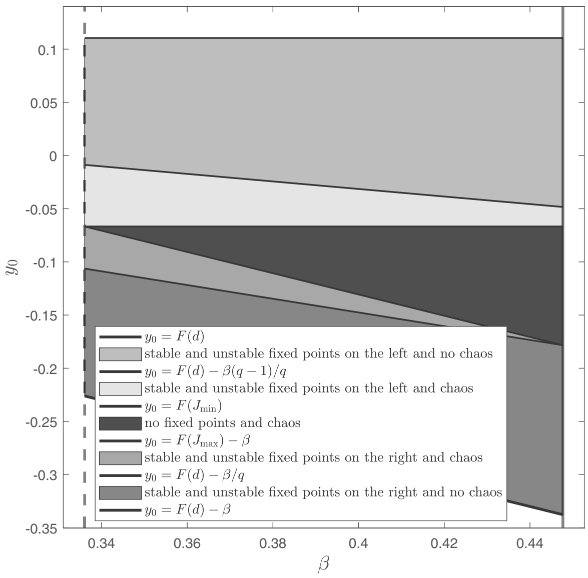

Fig. 4. Regions of different dynamics of the plCNV model.

classifying spike patterns generated by a neuron. Below we summarize some important observations that apply to both the 1D plCNV and nlCNV models. However, a similar interpretation holds also in the 2D case and can be formulated with respect to itineraries of the projection of 2D periodic orbits to their x-coordinate.

Firstly observe that, by definition, periodic orbits with rotation number $1/q$ or $(q-1)/q$ are always twist and have periodic itineraries of the form $LL\dots LR$ and $LRR\dots R$, respectively (with $q-1$ repetitions of the same corresponding symbol). Therefore, as follows from Corollary 1 (and independently from Corollary 2), if rational numbers of this form are in the rotation interval, then there are unique periodic orbits which realize them. Thus the spike trains corresponding to these rotation numbers are also unique.

Corollary 3. *Suppose that the number $1/q$ is in the rotation interval of the expanding Lorenz map describing the 1D CNV model. Then there is a unique initial voltage condition x_0 (in the invariant interval) which gives rise to the periodic spiking pattern where $q-1$ time instances of the increase in the membrane voltage x are followed by one rapid decrease in the membrane voltage. The period of this periodic pattern consists of q time instances, with exactly one spike in each period.*

The same holds for the rotation number $(q-1)/q$, with the difference that now an increase in the voltage per one period of length q occurs for 1 time unit and is followed by $q-1$ time instances of voltage decrease.

Note that a periodic orbit with the rotation number $1/q$ corresponds to the situation where the consecutive increase in the membrane potential (depolarization) for $q-1$ time instances is followed by quite a rapid decrease (repolarization) which indeed reminds a spike. In this case all the spikes along the same spike train look exactly the same (in particular they have the same amplitude).

On the other hand, the spiking pattern corresponding to the rotation number $(q-1)/q$ consists of one rapid increase of the membrane voltage followed by consecutive $q-1$ time instances in repolarization. Thus in this case the decrease in the membrane voltage following a spike is milder than in the case of rotation number $1/q$.

Similarly, one can obtain easily the following result.

Corollary 4.

- *A periodic orbit with rotation number $1/q$ results in a q-periodic voltage-train with $q-p$ time instances at which the membrane voltage increases and p time instances of the decrease in the voltage. Moreover, if p/q (p and q coprime) is in the rotation interval (of the expanding Lorenz map of the plCNV or nlCNV model), then there are at most $\binom{q}{p}/q$ corresponding periodic orbits, among which exactly one periodic orbit is twist but each of these orbits has a different itinerary.*
- *Identifying itineraries with periodic spike-trains produced by the corresponding periodic orbit, we conclude that every periodic orbit of the expanding Lorenz map of plCNV or nlCNV model gives rise to the unique spike-train.*

Thus the rotation number ϱ of a given periodic orbit together with its itinerary is directly related to the periodic spiking pattern generated by this orbit. In particular, one might assume than any occurrence of the word LR in the orbit itinerary might be interpreted as a spike. However, here (except for the case $\varrho = 1/q$ and $\varrho = (q-1)/q$) the amplitude of the spike might be different for each occurrence of the word LR in the q-length periodic sequence of L and R symbols (see Example 2 and Fig. 5). Thus, hypothetically (depending on the purpose) one can consider introducing some threshold value θ such that "spikes" with the amplitude above this threshold indeed count as spikes whereas "spikes" with the amplitude below θ might be rather interpreted as subthreshold oscillations. With this interpretation, knowing the itinerary of the periodic orbit might not be enough to fully describe the resulting spiking pattern as one might need a precise location of points of the orbit, not only the continuity interval in which they appear.

Corollary 5. *If p/q (p and q coprime) is in the rotation interval of the expanding Lorenz map of the 1D plCNV or nlCNV model then there exists a unique q-periodic spiking pattern of the form $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_q)$ (where $\tilde{x}_i \in \{L, R\}$) such that $\tilde{x}_i = L$ if and only if $1 + (i-1)p \mod q \leq q-p$ (with L-symbols corresponding to the increase in the neuron's membrane potential and R with the decrease).*

If the occurrence of the word LR is interpreted as a spike, then this spike train features n spikes in each q-cycle, where

$$n := \#\{1 < j < q \mid 1 + (j-1)p \mod q \le q - p \\ \text{and } 1 + jp \mod q > q - p\}$$

(with the convention that $q \bmod q = q$).

Proof. It is not hard to guess that the spike train concerned in the above corollary corresponds in fact to a twist periodic orbit with rotation number p/q. Such a twist periodic orbit exists and is unique for the expanding Lorenz map. Thus let $\{x_1 < x_2 < \ldots < x_q\}$ denote the set of points of this orbit arranged in the increasing order of their values. If $(z_1, z_2, \ldots, z_q)$ is the ordering of the orbit such that $G(z_i) = z_{i+1}$ (where G is the map of the 1D model), which corresponds to the spike train observed, then from the definition of the twist periodic orbit we have that

$$\begin{aligned} z_1 &:= x_1 \in I_L, \\ z_2 &= G(z_1) = G(x_1) = x_{1+p \mod q}, \\ &\vdots \\ z_i &= G(z_{i-1}) = x_{1+(i-1)p \mod q}, \\ &\vdots \\ z_q &= G(z_{q-1}) = x_{1+(q-1)p \mod q} = x_{q-p+1} \in I_R, \end{aligned}$$

where

$$z_i \in I_L \iff (1 + (i-1)p) \mod q \le q - p$$

as follows from the ordering $\{x_1 < x_2 < \ldots < x_q\}$ and the fact that the orbit with rotation number p/q has exactly p points in the right continuity interval I_R and $q - p$ points in the left interval I_L. The corresponding spike train $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_q)$ can be encoded with the sequence of symbols $\{L, R\}^q$ where $\tilde{x}_i = L$ if and only if $1 + (j-1)p \mod q \le q - p$ and the rest of the statement follows. ■

Example 1. Under the assumptions of Corollary 5, assume that $p/q = 4/5$ is in the rotation interval of the corresponding 1D model. Then there is a unique 5-periodic spike train of the form $LRRRR$. The spike-train, visualized in Figs. 5(a) and (b) together with the associated periodic orbit, features one spike in each cycle of length 5. ♦

Example 2. Similarly, assume that $p/q = 7/9$. Then there is a unique 9-periodic spike train of the form $LRRRLRRRR$. The spike-train features two spikes in each cycle of length 9 and these spikes are of different amplitude, as seen in Fig. 5(d). The corresponding periodic trajectory is depicted in Fig. 5(c). ♦

10. Conclusions

Our work, although devoted to a particular neuron model, might serve also as a review of the theory of Lorenz-like maps for which we have provided a couple of noteworthy extensions. Namely, for expanding Lorenz maps we have justified the expansiveness (Proposition 1), the uniqueness of periodic orbits with a given itinerary (Proposition 4), existence and uniqueness of twist periodic orbits with a given rotation number (Corollary 1), an upper estimate of the number of periodic orbits with the same rotation number (Corollary 2), the equivalence of transitivity and Devaney chaos on the whole interval (Proposition 5) and precise conditions for Devaney chaos (Theorem 7).

All these results can be applied to both linear and nonlinear CNV models. On the other hand, we also proved more particular results on β-transformations that apply in the linear case (Theorems 10 and 11 as well as Lemmas 1 and 2).

These mathematical results can be naturally applied for the study of the CNV model, mainly in its 1D reduction but partially also in its full 2D version, as we have shown in Sections 8 and 9. It has been rigorously shown how the theory of Lorenz-like maps and β transformation can be used for studying the CNV model, especially by identifying regions of chaotic and periodic behaviour in the model.

In particular, we have established the direct connection between periodic spiking patterns and periodic orbits of the voltage map in the CNV model and next, with the use of very recent findings on Farey-Lorenz permutations for Lorenz maps, we have provided a combinatorial description of the periodic spiking patterns. The rotation interval and the corresponding set of *fupos* might be viewed here as a complexity measure of the system under study.

The CNV model is useful in modelling and validating results of clinical relevance as exemplified in Introduction. Furthermore, since Lorenz-like maps appear also in hybrid neural models (see, e.g., Rubin *et al.*, 2017) and in various other models of physiological and biological phenomena (see, e.g., the work of Derks *et al.* (2021) and the references therein), as well as in numerous other applications, the methods presented and developed in this work might be adapted in multiple fields.

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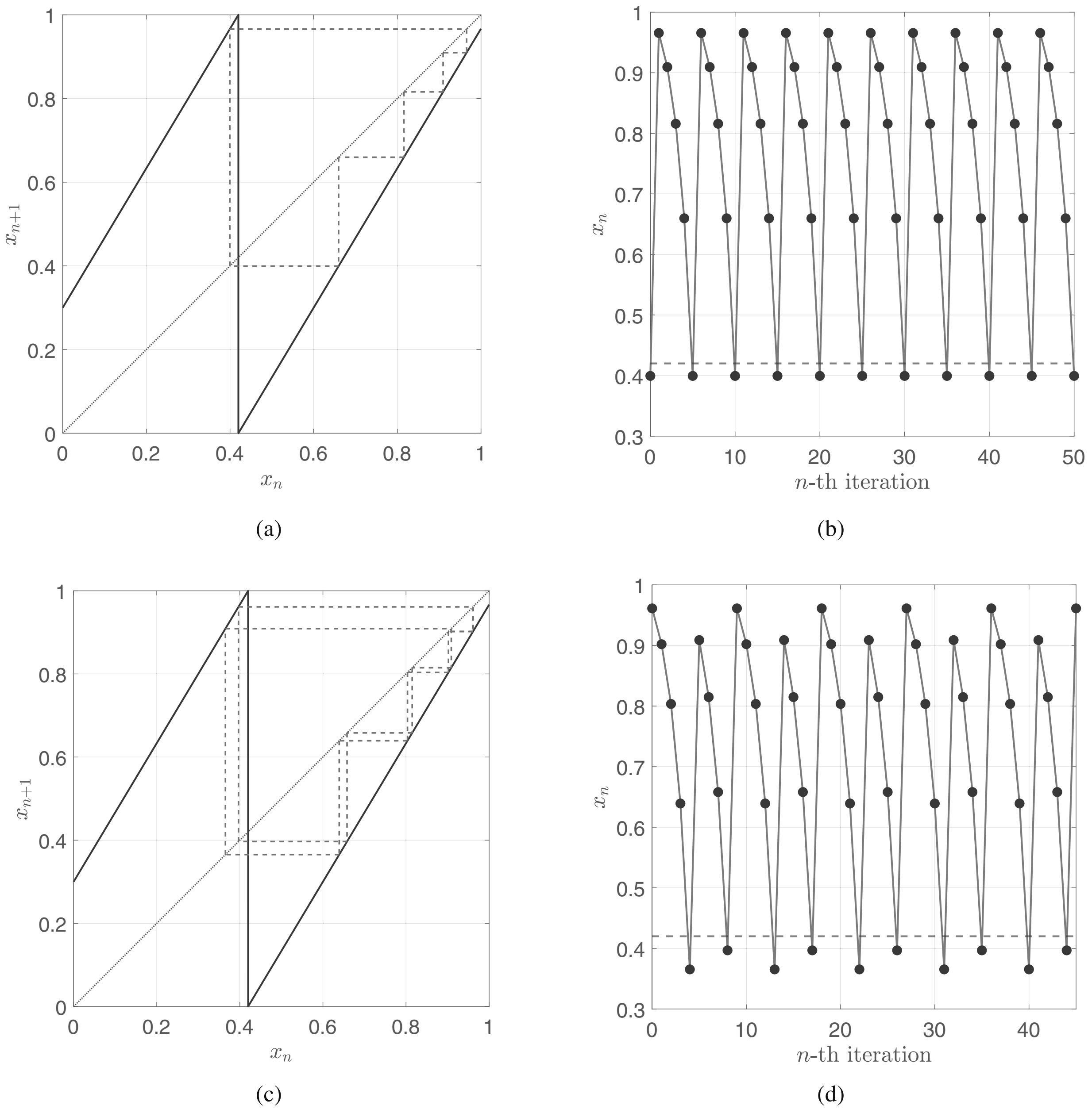

Fig. 5. Relation between the rotation number and the spike pattern: periodic orbit of the 1D model with rotation number $4/5$ (a) and the corresponding voltage trace (b), periodic orbit with rotation number $7/9$ (c) and corresponding voltage trace (d).

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